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Mathematical Statistical Physics: Assignment 1

Problem 1: *In high dimension, oranges are mostly peel.* (hand in, 26 points)
Show that for all $\varepsilon, \delta \in (0,1)$ there is $d_0 \in \mathbb{N}$ such that, for all $d > d_0$, a fraction of at least $1-\varepsilon$ of the volume of the unit ball in $\mathbb{R}^d$ is contained in the shell of thickness δ underneath the surface.

Problem 2: *Normalization of the Gaussian* (don't hand in)
Show that for all $\mu \in \mathbb{R}$ and $\sigma > 0$,

$$\int_{-\infty}^{+\infty} dx \, \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = 1\,.$$

Problem 3: *Gamma function* (hand in, 24 points)
Show that the Gamma function, defined on $(0,\infty)$ by $\Gamma(\alpha) = \int_0^\infty dt\, t^{\alpha-1} e^{-t}$, has the following properties.

(a) $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$.

(b) $\Gamma(1) = 1$. Thus, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

(c) $\Gamma(1/2) = \sqrt{\pi}$ (Hint: substitute $s = \sqrt{t}$). Thus, $\Gamma(n+1/2) = \dfrac{(2n)!\sqrt{\pi}}{4^n n!}$.

Problem 4: *Spherical coordinates in* $\mathbb{R}^d$ (hand in, 50 points)
They are defined by

$$\begin{aligned} x_1 &= r\cos\phi_1 \\ x_2 &= r\sin\phi_1\cos\phi_2 \\ x_3 &= r\sin\phi_1\sin\phi_2\cos\phi_3 \\ &\ldots \\ x_{d-1} &= r\sin\phi_1\cdots\sin\phi_{d-2}\cos\phi_{d-1} \\ x_d &= r\sin\phi_1\cdots\sin\phi_{d-1} \end{aligned} \tag{1}$$

with $r \in [0,\infty)$, $\phi_1,\ldots,\phi_{d-2} \in [0,\pi]$, and $\phi_{d-1} \in [0,2\pi)$.

(a) Show that for fixed $r > 0$, the image of the ϕ coordinates is the sphere of radius r, $\mathbb{S}_r^{d-1} = \big\{(x_1,\ldots,x_d) \in \mathbb{R}^d : x_1^2 + \ldots + x_d^2 = r^2\big\}$.

(b) Show that the Jacobian determinant of the coordinate transformation (1) is

$$J = r^{d-1} \sin^{d-2}\phi_1 \sin^{d-3}\phi_2 \cdots \sin\phi_{d-2}\,.$$

(In other words, the ($d-1$-dimensional) area dA of a surface element is

$$dA = r^{d-1} \sin^{d-2}\phi_1 \sin^{d-3}\phi_2 \cdots \sin\phi_{d-2}\, d\phi_1\, d\phi_2 \cdots d\phi_{d-1}\,,$$

and the (d-dimensional) volume of a volume element is $dV = dr\, dA$.)

(c) Show that the area of $\mathbb{S}^{d-1}_r$ is given by

$$A = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1}\,, \tag{2}$$

where Γ is the Gamma function, and the volume of the ball $B_r \subset \mathbb{R}^d$ by

$$V = \frac{\pi^{d/2}}{\Gamma(1+d/2)} r^{d}\,. \tag{3}$$

Hint: Use without proof that $\displaystyle\int_0^\pi d\phi\, \sin^k\phi = \sqrt{\pi}\,\Gamma\big(\tfrac{k+1}{2}\big)/\Gamma\big(\tfrac{k+2}{2}\big)$.

Problem 5: *Non-global solution* (don't hand in)
Verify that the trajectory (2.10) in the lecture notes is a solution of the equation of motion (2.1).

Problem 6: *Variance of a random variable* (don't hand in)
Let $\mathbb{E}X$ denote the expectation value of the real random variable X. The variance of X is defined as $\operatorname{Var} X = \mathbb{E}\big[(X - \mathbb{E}X)^2\big]$. Show that $\operatorname{Var} X = \mathbb{E}(X^2) - (\mathbb{E}X)^2$.

Hand in: By 8:15am on Tuesday, April 26, 2022 via `urm.math.uni-tuebingen.de`