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Mathematical Statistical Physics: Assignment 2

Problem 7: *Heat capacity* (don't hand in)
When a physical body is in thermal equilibrium, its temperature is spatially constant. According to the *zeroth law of thermodynamics*, an isolated physical body will sooner or later reach thermal equilibrium and remain there. The *heat* Q of a body is the energy of the thermal motion of its atoms. The heat capacity C of a physical body in thermal equilibrium at temperature T is defined to be dQ/dT, where dQ is the amount of energy that must be supplied to the body to increase its temperature by dT.

(a) When two separated systems in thermal equilibrium at the same temperature are brought into thermal contact, the composite is also in thermal equilibrium. When their thermal contact is interrupted, each is still in thermal equilibrium at the same temperature as before. Explain why their heat capacities add, $C(T) = C_1(T)+C_2(T)$. ("Heat capacity is an extensive property.")

(b) Consider two systems $\mathscr{S}_1, \mathscr{S}_2$ with the property $Q_i = C_i T_i$ $(i = 1, 2)$ and assume that Q_i is the only contribution to the total energy of $\mathscr{S}_i$. Suppose that system $\mathscr{S}_1$ is isolated and in thermal equilibrium at temperature T_1, $\mathscr{S}_2$ likewise at $T_2 < T_1$. Now we bring $\mathscr{S}_1$ and $\mathscr{S}_2$ in thermal contact and wait for $\mathscr{S}_1 \cup \mathscr{S}_2$ to reach thermal equilibrium. Determine the resulting temperature T_{12} of $\mathscr{S}_1 \cup \mathscr{S}_2$.

Problem 8: *Conservation laws* (hand in, 33 points)
Prove Proposition 1, i.e., the conservation laws of energy, momentum, and angular momentum, assuming the equation of motion (2.1).

Problem 9: *Elastic collision of two billiard balls in* $\mathbb{R}^3$ (don't hand in)
Prior to the collision, the two balls of radius $a > 0$ and mass $m > 0$ move at constant velocities. Let the locations at the time of collision be $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ with $|\boldsymbol{q}_1 - \boldsymbol{q}_2| = 2a$, the momenta prior to the collision $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ with $(\boldsymbol{p}_2 - \boldsymbol{p}_1)\cdot(\boldsymbol{q}_2 - \boldsymbol{q}_1) < 0$ (they move towards each other, not away), and let $\boldsymbol{\omega}$ be the unit vector from ball 1 to ball 2, $\boldsymbol{\omega} = (\boldsymbol{q}_2 - \boldsymbol{q}_1)/2a$. Assume that the balls cannot spin. Show that conservation of energy, momentum, and angular momentum allow only two possibilities for the momenta $\boldsymbol{p}_1'$ and $\boldsymbol{p}_2'$ after the collision: Either $\boldsymbol{p}_1' = \boldsymbol{p}_1$ and $\boldsymbol{p}_2' = \boldsymbol{p}_2$ (which is impossible if the balls cannot pass through each other), or the $\boldsymbol{\omega}$ components of the $\boldsymbol{p}_k$ get exchanged while the components perpendicular to $\boldsymbol{\omega}$ remain unchanged. Show further that the latter case means

$$\boldsymbol{p}_1' = \boldsymbol{p}_1 - [(\boldsymbol{p}_1 - \boldsymbol{p}_2)\cdot\boldsymbol{\omega}]\boldsymbol{\omega}\,, \quad \boldsymbol{p}_2' = \boldsymbol{p}_2 + [(\boldsymbol{p}_1 - \boldsymbol{p}_2)\cdot\boldsymbol{\omega}]\boldsymbol{\omega}\,.$$

("Elastic" means conserving energy. Here, $E = \boldsymbol{p}_1^2/2m + \boldsymbol{p}_2^2/2m$ and $\boldsymbol{L} = \boldsymbol{q}_1 \times \boldsymbol{p}_1 + \boldsymbol{q}_2 \times \boldsymbol{p}_2$.)

Problem 10: *Distance of measures* (hand in, 34 points)
The *total variation distance* of two finite measures μ, ν on the same σ-algebra $\mathscr{A}$ is defined by

$$d(\mu,\nu) = \sup_{A\in\mathscr{A}} \big(\mu(A)-\nu(A)\big) + \sup_{A\in\mathscr{A}} \big(\nu(A)-\mu(A)\big)\,.$$

Now let μ, ν be probability measures.
(a) Show that $d(\mu,\nu) = 2\sup_{A\in\mathscr{A}}\big|\mu(A)-\nu(A)\big|$.
(*Hint*: $\nu(A)-\mu(A) = \mu(A^c)-\nu(A^c)$.)
(b) Show that if μ and ν possess densities f and g relative to the measure λ on $\mathscr{A}$, then the total variation distance coincides with the L^1 norm of $f-g$,

$$d(\mu,\nu) = \int \lambda(dx)\,|f(x)-g(x)|\,.$$

Problem 11: *Another way to compute the area of* $\mathbb{S}_1^{d-1}$ (hand in, 33 points)
The integral

$$\int_{\mathbb{R}^d} d^d\boldsymbol{x}\; e^{-\boldsymbol{x}^2} \tag{1}$$

can be computed in two ways: as a product of d 1-dimensional integrals (whose values we know), or in spherical coordinates (where the angle integrals yield the area A_d of $\mathbb{S}_1^{d-1}$). Exploit this to show that

$$A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,. \tag{2}$$

Hand in: By 8:15am on Tuesday, May 3, 2022 via `urm.math.uni-tuebingen.de`