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Mathematical Statistical Physics: Assignment 5

Problem 21: *Types of convergence of measures* (don't hand in)
(a) Compute the total variation distance between a continuous distribution and the empirical distribution of N i.i.d. realizations.

(b) Which one is stronger, convergence in total variation or convergence of the histogram (i.e., of the coarse grained distribution relative to a finite partition)? Show that one implies the other.

Problem 22: *Poincaré recurrence* (hand in, 35 points)
Let $\Omega = \mathbb{S}^1_1$ be the unit circle in $\mathbb{R}^2 = \mathbb{C}$, and let $T : \Omega \to \Omega$ be multiplication by $e^{i\alpha}$. Use the Poincaré recurrence theorem to show:

(a) $\forall \delta > 0 : \exists n \in \mathbb{N} : \forall x \in \Omega : \ d(x, T^n x) < \delta$,
where d is the distance in arc length along the circle.

(b) For $\alpha \notin \pi\mathbb{Q}$ and every $x \in \Omega$, the set $T^{\mathbb{N}}x$ is dense in Ω.

Problem 23: *Dense trajectory* (hand in, 30 points)
Let Ω be the 2-dimensional torus and φ_1 and φ_2 the angular coordinates on it (longitude and latitude). For given constants $\alpha_1, \alpha_2 \in \mathbb{R}$, consider the ODE

$$\frac{d\varphi_1}{dt} = \alpha_1 \,, \quad \frac{d\varphi_2}{dt} = \alpha_2 \,.$$

(a) Give an explicit formula for the flow map: $T^t(\varphi_1, \varphi_2) = \,?$

(b) Use Problem 22 to show that if $\alpha_2 \neq 0$ and $\alpha_1/\alpha_2 \notin \mathbb{Q}$, then the curve $t \mapsto T^t(\varphi_1, \varphi_2)$ is dense on the torus.

Problem 24: *Dense trajectory in higher dimension* (don't hand in)
Consider the corresponding situation on the n-dimensional torus $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$:

$$\frac{d\varphi_i}{dt} = \alpha_i \,, \quad i = 1, \ldots, n \,.$$

Under which condition on $(\alpha_1, \ldots, \alpha_n)$ is the curve $t \mapsto T^t(\varphi_1, \ldots, \varphi_n)$ dense on the torus?

Problem 25: *Recurrence times* (don't hand in)
In order to estimate the order of magnitude of realistic recurrence times, we reason as follows. An ideal gas comprising $N = 10^{23}$ particles (or a gas of 10^{23} hard spheres, the difference does not matter) in a box Λ starts in such a phase point x_0 that all particles are located in the left half Λ_L of the box; apart from that, let x_0 be typical of energy $E = N\bar{e}$; i.e., take x_0 to be a typical element of $M_L = \Gamma_E \cap (\Lambda_L^N \times \mathbb{R}^{3N})$. We want to know how long it takes, after $x(t)$ has left M_L, until $x(t)$ returns to M_L.

(a) Determine $\mu_E(M_L)$.

(b) Think of Γ_E as partitioned into cells $C_1, \ldots, C_r$ of equal volume (i.e., of equal measure μ_E), of which M_L is one. Assume that every cell gets traversed in time τ, and that the trajectory $x(t)$ visits all cells in a random-looking order. How many years will pass before the return to M_L if $\tau = 10$ s? If $\tau = 10^{-20}$ s?

Problem 26: *Ergodicity on a finite set* (hand in, 35 points)
Let Ω be a finite set, $\#\Omega = n$. A dynamical system in discrete time means a bijection $T : \Omega \to \Omega$.
(a) Show that T preserves the uniform measure $\mathbb{P}(A) = \#A/n$.
(b) Show that $\mathbb{P}$ is the only probability measure preserved by every bijection.
(c) How many dynamical systems on Ω exist?
(d) What should it mean here for T to be ergodic?
(e) What are the ergodic components of a non-ergodic T?
(f) How many dynamical systems on Ω are ergodic?
(g) Determine the probability that a randomly chosen dynamical system on Ω is ergodic.
(h) The recurrence time of $\omega \in \Omega$ for T is defined as the smallest $t \in \mathbb{N}$ with $T^t\omega = \omega$. Determine the recurrence time for ergodic T.
(i) Determine the average recurrence time for random T (ergodic or not) and random ω.

Hand in: By 8:15am on Tuesday, May 24, 2022 via `urm.math.uni-tuebingen.de`