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Mathematical Statistical Physics: Assignment 6

Problem 27: *Refrigerator* (don't hand in)
Use the second law of thermodynamics to show that a refrigerator necessarily consumes energy rather than generating energy. (We might have thought that the energy removed from the content of the refrigerator is available afterwards.)

Instructions. Let T_{in} be the temperature inside the refrigerator, T_{out} the one outside, Q_{in} the heat energy inside, Q_{out} the one outside, and δW the usable energy provided by the refrigerator (negative if it consumes energy) while it adds the energy $\delta Q_{\text{in}} < 0$ to the content and δQ_{out} to the outside. Use the Clausius relation $\delta S_i = \delta Q_i / T_i$, $i = \text{in}, \text{out}$ to show that $\delta W < 0$.

Problem 28: *Entropy in thermal equilibrium* (hand in, 30 points)
Compute the entropy $S(E, V, N)$ of the thermal equilibrium state of an ideal mono-atomic gas from Boltzmann's formula $S(\text{eq}) = k \log \Omega(E)$.

Instructions. We have already found the relations

$$\operatorname{vol} \Gamma_{\leq E} = \frac{1}{N!} V^N \, V_{3N} (2mE)^{3N/2} \tag{1}$$

$$V_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)} \tag{2}$$

$$\Omega(E) = \frac{d}{dE} \operatorname{vol} \Gamma_{\leq E} \, . \tag{3}$$

Use Stirling's formula

$$\Gamma(x+1) = \sqrt{2\pi x} \, e^{-x} x^x (1 + o(1)) \quad \text{as } x \to \infty \tag{4}$$

and $n! = \Gamma(n+1)$. Set $E = Ne$ and $V = Nv$ with constants e, v, sort terms by orders $O(N \log N), O(N), O(\log N), \ldots$, and give the leading order terms as the answer.

Problem 29: *Concave function* (hand in, 20 points)
Verify that the function

$$S(E, V, N) = kN \log \frac{V}{N v_0} + \tfrac{3}{2} kN \log \frac{E}{N e_0}$$

is concave on $(0, \infty)^3$. Use without proof that a C^2 function is concave if and only if its Hessian is everywhere negative semi-definite. (*Hint*: The matrix is singular, and on a certain 2d subspace it is negative definite.)

Problem 30: *Scattering cross section for billiard balls* (hand in, 30 points)
When two billiard balls of radius a and momenta $\boldsymbol{p}_1, \boldsymbol{p}_2$ collide, the resulting (outgoing) momenta $\boldsymbol{p}_1', \boldsymbol{p}_2'$ depend on the displacement vector $\boldsymbol{\omega} = (\boldsymbol{q}_2 - \boldsymbol{q}_1)/2a \in \mathbb{S}^2_1$ at the time of the collision:

$$\boldsymbol{p}_1' = \boldsymbol{p}_1 - [(\boldsymbol{p}_1 - \boldsymbol{p}_2)\cdot\boldsymbol{\omega}]\boldsymbol{\omega}\,, \qquad \boldsymbol{p}_2' = \boldsymbol{p}_2 + [(\boldsymbol{p}_1 - \boldsymbol{p}_2)\cdot\boldsymbol{\omega}]\boldsymbol{\omega}\,. \tag{5}$$

We consider *random* collisions and want to characterize the probability distribution of $\boldsymbol{p}_1', \boldsymbol{p}_2'$ for given $\boldsymbol{p}_1, \boldsymbol{p}_2$ by determining that of $\boldsymbol{\omega}$. To this end, we suppose that $\boldsymbol{p}_2 = \boldsymbol{0}$ (as can be arranged via a Galilean transformation), $\boldsymbol{q}_2 = \boldsymbol{0}$, and $\boldsymbol{p}_1 = p_1 \boldsymbol{e}_x$ (via translation and rotation). It is reasonable to assume that the y- and z-components of $\boldsymbol{q}_1$ are uniformly distributed on the disc of radius $2a$ around the origin in the yz-plane (given that a collision occurs at all); the polar coordinates r and φ of (y, z) are called the collision parameters.

(a) Express $\boldsymbol{q}_1$ and $\boldsymbol{\omega}$ as functions of r and φ.
(b) Show that $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ has probability density proportional to $1_{\omega_x<0}\,|\omega_x|$ relative to the uniform measure $u(d^2\boldsymbol{\omega})$ on the sphere.
(c) Explain why, for arbitrary $\boldsymbol{p}_1, \boldsymbol{p}_2$, the probability distribution of $\boldsymbol{\omega}$ is proportional to $1_{\boldsymbol{\omega}\cdot(\boldsymbol{p}_1-\boldsymbol{p}_2)<0}\,\big|\boldsymbol{\omega}\cdot(\boldsymbol{p}_1 - \boldsymbol{p}_2)\big|\,d^2\boldsymbol{\omega}$.

Problem 31: *Gibbs entropy* (hand in, 20 points)
The Gibbs entropy of a probability density function ρ on phase space $\Gamma = \mathbb{R}^d$ is defined by

$$S_{\text{Gibbs}}(\rho) = -k \int_\Gamma dx\, \rho(x)\, \log \rho(x) \tag{6}$$

with the convention $0 \log 0 := 0$. Suppose $M : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism with Jacobian determinant $|\det DM(x)| = 1$ at all $x \in \mathbb{R}^d$ (so M preserves volumes), and suppose that the point X_0 in $\mathbb{R}^d$ is chosen randomly with (smooth) probability density $\rho_0 : \mathbb{R}^d \to [0, \infty)$. Use the transformation formula for integrals to show that

(a) the random point $Y = M(X_0)$ has density $\rho_1(y) = \rho_0(M^{-1}(y))$.

(b) $S_{\text{Gibbs}}(\rho_1) = S_{\text{Gibbs}}(\rho_0)$.

Remark. As a consequence, for a Hamiltonian system on $\mathbb{R}^d$ such that for each $t \in \mathbb{R}$ the flow map T^t is a diffeomorphism $\mathbb{R}^d \to \mathbb{R}^d$, if ρ_t is the density of $X_t = T^t(X_0)$ then, since T^t has Jacobian determinant 1 (Liouville's theorem), the Gibbs entropy never changes with time.

Hand in: By 8:15am on Tuesday, May 31, 2022 via `urm.math.uni-tuebingen.de`