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Mathematical Statistical Physics: Assignment 8

Problem 36: *Properties of the collision transformation* (hand in, 50 points)
At a collision of two billiard balls with collision parameter $\boldsymbol{\omega} = (\boldsymbol{q}_2 - \boldsymbol{q}_1)/2a$, the velocities change from $\boldsymbol{v} = \boldsymbol{v}_1$ and $\boldsymbol{v}_* = \boldsymbol{v}_2$ to

$$\boldsymbol{v}' = \boldsymbol{v} - [(\boldsymbol{v} - \boldsymbol{v}_*) \cdot \boldsymbol{\omega}]\boldsymbol{\omega} \tag{1}$$

$$\boldsymbol{v}'_* = \boldsymbol{v}_* + [(\boldsymbol{v} - \boldsymbol{v}_*) \cdot \boldsymbol{\omega}]\boldsymbol{\omega}\,. \tag{2}$$

Let $R_{\boldsymbol{\omega}}$ be the linear mapping $\mathbb{R}^6 \to \mathbb{R}^6$ with $R_{\boldsymbol{\omega}}(\boldsymbol{v}, \boldsymbol{v}_*) = (\boldsymbol{v}', \boldsymbol{v}'_*)$. Show that

- **(a)** $R_{\boldsymbol{\omega}}$ is orthogonal, $R_{\boldsymbol{\omega}} \in O(6)$. (*Hint*: By the polarization identity $\boldsymbol{u} \cdot \boldsymbol{v} = \frac{1}{4}\big(|\boldsymbol{u} + \boldsymbol{v}|^2 - |\boldsymbol{u} - \boldsymbol{v}|^2\big)$, it suffices for orthogonality of a linear mapping A that $|A\boldsymbol{u}| = |\boldsymbol{u}|$ for all $\boldsymbol{u}$.)
- **(b)** $\det R_{\boldsymbol{\omega}} = -1$. (You may use without proof that the determinant of a block matrix $\big(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\big)$, where A, B, C, D all commute with each other, is[1] $\det(AD - BC)$.)
- **(c)** $R_{\boldsymbol{\omega}}^2 = I_6$
- **(d)** $R_{-\boldsymbol{\omega}} = R_{\boldsymbol{\omega}}$
- **(e)** $\boldsymbol{\omega} \cdot (\boldsymbol{v}' - \boldsymbol{v}'_*) = -\boldsymbol{\omega} \cdot (\boldsymbol{v} - \boldsymbol{v}_*)$.

Problem 37: *Boltzmann equation with external potential* (hand in, 25 points)
In an external potential V_1, the Boltzmann equation reads

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{q}} - \frac{1}{m}\nabla V_1(\boldsymbol{q}) \cdot \nabla_{\boldsymbol{v}}\right) f(\boldsymbol{q}, \boldsymbol{v}, t) = Q(\boldsymbol{q}, \boldsymbol{v}, t) \tag{3}$$

with the same collision term as given in the lectures,

$$Q(\boldsymbol{q}, \boldsymbol{v}, t) = \lambda \int_{\mathbb{R}^3} d^3\boldsymbol{v}_* \int_{\mathbb{S}^2} d^2\boldsymbol{\omega}\; 1_{\boldsymbol{\omega}\cdot(\boldsymbol{v}-\boldsymbol{v}_*)>0}\; \boldsymbol{\omega} \cdot (\boldsymbol{v} - \boldsymbol{v}_*) \times \Big[f(\boldsymbol{q}, \boldsymbol{v}', t)\, f(\boldsymbol{q}, \boldsymbol{v}'_*, t) - f(\boldsymbol{q}, \boldsymbol{v}, t)\, f(\boldsymbol{q}, \boldsymbol{v}_*, t)\Big]\,, \tag{4}$$

and boundary condition

$$f(\boldsymbol{q}, \boldsymbol{v}, t) = f\big(\boldsymbol{q}, \boldsymbol{v} - 2[\boldsymbol{v} \cdot \boldsymbol{n}]\boldsymbol{n}, t\big) \tag{5}$$

for $\boldsymbol{q} \in \partial\Lambda$ and $\boldsymbol{n} = \boldsymbol{n}(q)$ the outward unit normal vector. Show that the Maxwell-Boltzmann distribution is a stationary solution.

[1] J. R. Silvester: Determinants of Block Matrices. *The Mathematical Gazette* **84(501)**: 460–467 (2000)

Problem 38: *A class of solutions of the Boltzmann equation* (don't hand in)
Show that functions of the form

$$f_t(\boldsymbol{q},\boldsymbol{v}) = \exp\Big(A(\boldsymbol{q}) + \boldsymbol{B}_t(\boldsymbol{q})\cdot\boldsymbol{v} + C_t\boldsymbol{v}^2\Big) \tag{6}$$

with

$$A = A_1 + \boldsymbol{A}_2\cdot\boldsymbol{q} + C_3\boldsymbol{q}^2, \quad \boldsymbol{B} = \boldsymbol{B}_1 - \boldsymbol{A}_2 t - (2C_3 t + C_2)\boldsymbol{q} + \boldsymbol{B}_0\times\boldsymbol{q}, \quad C = C_1 + C_2 t + C_3 t^2$$

are solutions of the Boltzmann equation without external field and without walls.

Problem 39: *An inequality we use* (hand in, 25 points)
Let (as in the proof of Proposition 8)

$$g(z) = \begin{cases} z & \text{for } 0 \le z \le 1 \\ 1 & \text{for } z \ge 1\,. \end{cases}$$

Show that for all $x \ge 0$,

$$r(x) := x\log x + 1 - x \ge \tfrac{1}{4}\, g(|x-1|)\,|x-1|\,. \tag{7}$$

(*Hint*: As a start, use r' and r'' to show r is convex and to find its minimum.) For $x = f/M$ we obtain the relation (8.73) used in the proof of Proposition 8.

Hand in: By 8:15am on Tuesday, June 21, 2022 via `urm.math.uni-tuebingen.de`