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Mathematical Statistical Physics: Assignment 11

Problem 46: *Complex Gaussian in d dimensions* (hand in, 25 points)
The complex random variable X obeys the *complex Gaussian distribution* $\mathcal{N}_{\mathbb{C}}(z, \sigma^2)$ whenever $\operatorname{Re} X \sim \mathcal{N}(\operatorname{Re} z, \sigma^2/2)$, $\operatorname{Im} X \sim \mathcal{N}(\operatorname{Im} z, \sigma^2/2)$, and $\operatorname{Re} X$ and $\operatorname{Im} X$ are independent. The *Gaussian distribution* $\mathcal{N}_{\mathbb{C}^d}(\psi_0, C)$ *on* $\mathbb{C}^d$ has density of the form

$$f(\psi) = \mathscr{N}\, \exp\Big(-\langle \psi - \psi_0 | C^{-1} | \psi - \psi_0 \rangle\Big)\,, \tag{1}$$

where the complex $d \times d$ matrix C is self-adjoint and positive definite. Use what we know about the real Gaussian distribution to show for $\Psi \sim \mathcal{N}_{\mathbb{C}^d}(\psi_0, C)$ that
(a) $\mathscr{N} = \pi^{-d} (\det C)^{-1}$
(b) C is the complex covariance matrix, $C = \mathbb{E} |\Psi - \psi_0\rangle\langle \Psi - \psi_0|$ (*hint*: diagonalize)
(c) for every $\phi \in \mathbb{C}^d$ we have that $\langle \phi | \Psi \rangle$ obeys a complex Gaussian distribution.

Problem 47: *Conditional wave function* (hand in, 25 points)
For a given ONB $\{\phi_q\}$ of $\mathscr{H}_b$ and $\psi \in \mathbb{S}(\mathscr{H}_s \otimes \mathscr{H}_b)$, the *conditional wave function* ψ_s is the random vector in $\mathbb{S}(\mathscr{H}_s)$ given by

$$\psi_s = \mathscr{N}\, \langle \phi_Q | \psi \rangle_b$$

with partial inner product only in b, normalizing factor $\mathscr{N}$, and random Q,

$$\mathbb{P}(Q = q) = \| \langle \phi_q | \psi \rangle_b \|_s^2\,.$$

Show that the density matrix of the distribution μ_s of ψ_s is exactly the reduced density matrix of ψ, $\rho_{\mu_s} = \rho_s^\psi$ with $\rho_{\mu_s} = \mathbb{E} |\psi_s\rangle\langle \psi_s|$ and $\rho_s^\psi = \operatorname{tr}_b |\psi\rangle\langle\psi|$.

Problem 48: *Typicality of thermal equilibrium* (hand in, 25 points)
Let $0 < \varepsilon < \delta < 1$. Let $\mathscr{H}_{\mathrm{eq}}$ be a subspace of $\mathscr{H}_{\mathrm{mc}}$ with $\dim \mathscr{H}_{\mathrm{eq}} / \dim \mathscr{H}_{\mathrm{mc}} = 1 - \varepsilon$, let P_{eq} be the projection to $\mathscr{H}_{\mathrm{eq}}$, and let u_{mc} be the uniform distribution over $\mathbb{S}(\mathscr{H}_{\mathrm{mc}})$. Show for the set

$$G = \big\{ \psi \in \mathbb{S}(\mathscr{H}_{\mathrm{mc}}) : \langle \psi | P_{\mathrm{eq}} | \psi \rangle > 1 - \delta \big\}$$

that

$$u_{\mathrm{mc}}(G) > 1 - \frac{\varepsilon}{\delta}\,.$$

(As a consequence, if $\varepsilon \ll \delta \ll 1$, most wave functions lie in the set G of "equilibrium states.")

Hint: Determine the average of $\langle \psi | P_{\mathrm{eq}} | \psi \rangle$ on $\mathbb{S}(\mathscr{H}_{\mathrm{mc}})$. If a function $f \leq 1$ has average near 1, why does it have to be close to 1 at most points?

Problem 49: *Eigenfunctions in equilibrium* (hand in, 25 points)
Show that most eigenvectors of H in $\mathscr{H}_{\mathrm{mc}}$ (in fact at least the fraction $1 - \varepsilon/\delta$) lie in the set G from Problem 48. (Actually, that is true of every ONB. *Hint*: Proceed analogously to Problem 48.)

Hand in: By 8:15am on Tuesday, July 12, 2022 via `urm.math.uni-tuebingen.de`