Prof. Dr. R. Tumulka
Mathematisches Institut, Universität Tübingen

Summer semester 2022
July 2022

Practice Exam Math. Stat. Phys.

Instructions:

- These problems are similar in style to the exam problems. Don't hand them in, they will not be graded.
- The material covered by the exam comprises Sections 1–11.3 of the lecture notes and all homework assignments.
- The exam takes 90 minutes. It takes place on Thursday July 21, 2022, at 8:20-9:50 am in room N14.
- During the exam, it is not allowed to use books, electronic devices, or notes from before the exam.

Problem 1: (10 points) State Liouville's theorem.

Problem 2: (10 points) State the definition of ergodicity or an equivalent condition.

Problem 3: (10 points) Explain what kind of statement a "validity theorem" of the Boltzmann equation is.

Problem 4: (10 points) Explain Zermelo's "recurrence" objection to any attempted derivation of the second law from mechanics. (No need to answer the objection.)

Problem 5: (12 points) Consider a random point $\boldsymbol{X} = (X_1, \ldots, X_d)$ with uniform distribution u_R over the sphere $\mathbb{S}^{d-1}_R$ of radius $R = \sqrt{d}$ in $\mathbb{R}^d$. State mathematically what is meant by saying that
(a) "the marginal of u_R is Gaussian for large d";
(b) "the empirical distribution of $X_1, \ldots, X_d$ is typically Gaussian for large d."

Problem 6: (10 points) The integral

$$\mathscr{I} := \int_{\mathbb{R}^d} d^d\boldsymbol{x}\; e^{-\boldsymbol{x}^2} \tag{1}$$

can be computed in two ways: as a product of d 1-dimensional integrals (whose values we know), or in spherical coordinates (where the angle integrals yield the area A_d of $\mathbb{S}^{d-1}_1$). Exploit this to show that

$$A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,. \tag{2}$$

Problem 7: (14 points) Use the Poincaré recurrence theorem to show that for the discrete-time dynamical system on the unit circle $\mathbb{S}^1_1 = \{z \in \mathbb{C} : |z| = 1\}$ given by $Tz = e^{i\alpha}z$, the set $\{T^n 1 : n \in \mathbb{N}\}$ is dense on the circle if α/π is irrational.

Problem 8: (12 points) We want to show that P_+ defined by

$$P_+\psi(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N) = \frac{1}{N!} \sum_{\sigma \in S_N} \psi(\boldsymbol{q}_{\sigma(1)}, \ldots, \boldsymbol{q}_{\sigma(N)}) \tag{3}$$

is the orthogonal projection to the subspace of symmetric (bosonic) functions in $\mathscr{H} = L^2(\mathbb{R}^{3N})$. Proceed as follows:

(a) $P_+\psi$ is a symmetric function.

(b) If ψ is already symmetric, then $P_+\psi = \psi$.

(c) $P_+^2 = P_+$

(d) $P_+ : \mathscr{H} \to \mathscr{H}$ is self-adjoint.

Problem 9: (12 points) Consider the quantum system consisting of N non-interacting spins, $\mathscr{H} = (\mathbb{C}^2)^{\otimes N}$, $H = \sum_{i=1}^N I \otimes \cdots \otimes I \otimes H_i \otimes I \otimes \cdots \otimes I$ with H_i in the i-th place given by

$$H_i = \begin{pmatrix} 1/\sqrt{N} & 0 \\ 0 & -1/\sqrt{N} \end{pmatrix}.$$

Find the density of states $\Omega(E)$ in the limit $N \to \infty$.

—THE END—