phase space: $\mathbb{R}^{6N} =: \Gamma \ni x = (q, v)$

$q = (q_1 \cdots q_N)$, $v = (\underline{v}_1 \cdots \underline{v}_N)$

eq. of motion $m_k \frac{d^2 q_k}{dt^2} = \sum_{j \neq k} \underset{\substack{\uparrow \\ G m_j m_k - \frac{e_j e_k}{4\pi\varepsilon_0}}}{\text{const.}_{jk}} \frac{q_j - q_k}{|q_j - q_k|^3}$ (2.1)

PL thm: $\frac{dx}{dt} = F(x, t)$ (2.6)

$\frac{dq_k}{dt} = v_k$

$\frac{dv_k}{dt} = \frac{1}{m_k} \text{force}_k$

$F: Q_{\neq} \times \mathbb{R}^{3N+1} \xrightarrow{n = 6N} \mathbb{R}^{6N}$

$Q_{\neq} = \{ q \in \mathbb{R}^{3N} : q_j \neq q_k \ \forall j \neq k \}$

$3(N-1)$

$\frac{N(N-1)}{2}$

F not Lipschitz

$$\left.\begin{array}{l}\Omega \text{ compact} \subseteq \Omega_{\neq} \times \mathbb{R}^{3N+1} \\ \text{convex}\end{array}\right\} \Rightarrow F \text{ Lipschitz on } \Omega.$$

global existence: $t \mapsto x(t)$ $\forall t \in \mathbb{R}$, i.e. $x : \mathbb{R} \to \Gamma$

maximal solution: $I_{max} = (t_-, t_+) \subseteq \mathbb{R}$

$x_{max} : I_{max} \to \Gamma$ sol. to (2.6) $t_-, t_+ \in \mathbb{R} \cup \{\infty, -\infty\}$

s.t. $x_{max}(t_0) = x_0$, $t_0 \in I_{max}$.

<u>Corollary</u> If $F \in C^1(\Omega)$, $\Omega \subseteq \mathbb{R}^{n+1}$ open, then $\forall (x_0, t_0) \in \Omega$ $\exists_1$ max. sol. x_{max} of (2.6).

If $t_\pm \neq \pm\infty$ then either $\limsup_{t \to t_\pm} |x(t)| = \infty$

(blowup in finite time)

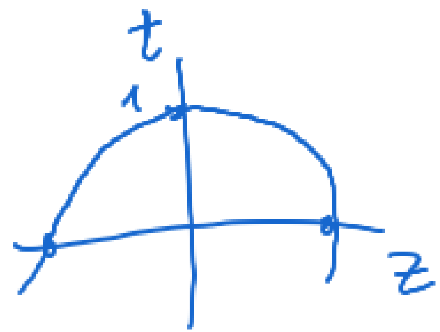
or $\lim_{t \to t_\pm} x(t) \in \partial\Omega$ (i.e. leaves Ω)

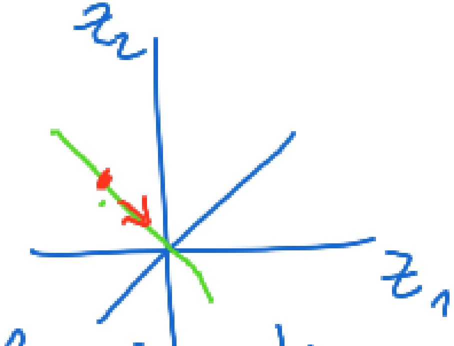
bdry

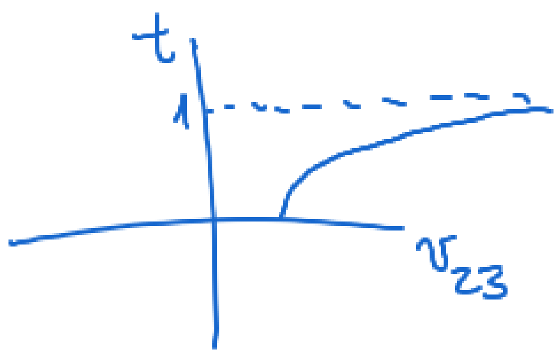
Ex (2.1), $N=2$, $e_1=0=e_2$, $m_1=m_2>0$.

$$(2.10)\quad q_1(t)=\left(0,0,(1-t)^{2/3}R\right),\quad q_2=\left(0,0,-(1-t)^{2/3}R\right)$$

$$R=\frac{1}{2}(9Gm)^{1/3}$$







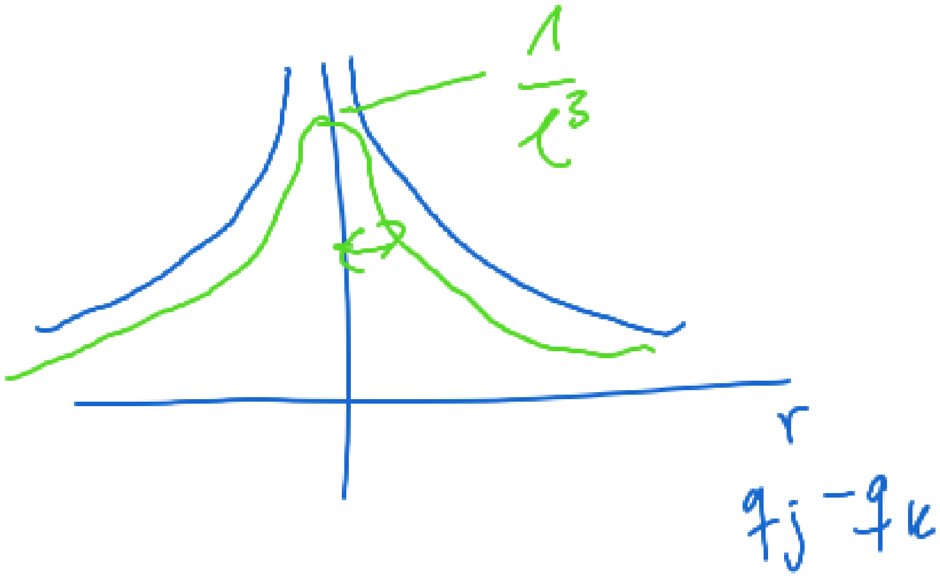
Hypothesis, Conjecture:

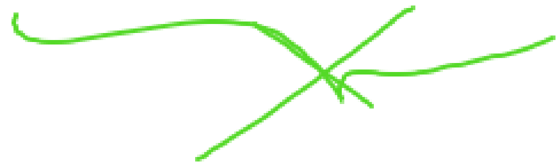
Almost all sol.s of (2.1) exist globally.

= all except a set of $\mathrm{vol}_{6N}=0$.

<u>Ex</u> $$\frac{q_j - q_k}{|q_j - q_k|^3} \rightsquigarrow \frac{q_j - q_k}{\left(|q_j - q_k|^2 + \ell^2\right)^{3/2}}$$

$0 < \ell \ll 1$





$$\frac{1}{\left(\;\_\;\;\_\;\_\;\right)^{3/2}}$$

Properties of the theory

1) time reversal invariance:

$t \mapsto q(t)$ sol. of (2.1)

$\Rightarrow$ $t \mapsto q(-t) = \tilde{q}(t)$ time reverse is a sol. as well.

$\tilde{q}(0) = q(0), \quad \tilde{v}(0) = -v(0)$

microscopic reversibility

macroscopic irreversibility

2nd law of thermodynamics: Loschmidt's paradox.

2) Conserved quantities

$\underline{x}^2 = \underline{x} \cdot \underline{x} = |\underline{x}|^2$

<u>Def</u> energy $E := \sum_{k=1}^{N} \frac{m_k}{2} \underline{v}_k^2 - \sum_{j<k} \mathrm{const.}_{jk} \frac{1}{|q_j - q_k|}$

momentum $\underline{p} := \sum_{k=1}^{N} m_k \underline{v}_k$

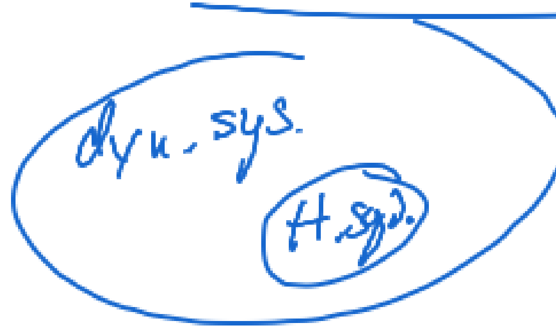
angular momentum $\underline{L} := \sum_{k=1}^{N} m_k q_k \times \underline{v}_k$

<u>Prop</u> E, $\underline{p}$, $\underline{L}$ are conserved.

Hamiltonian systems

dynamical system = ODE

$$\frac{dx}{dt} = F(x,t)$$



Ex $\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 = 1$.

F = time-dep. vector field

"$F = \nabla H$", more precisely: $x = (q, p)$,

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

symplectic form: $\quad \omega : V \times V \to \mathbb{R}$

$V = \mathbb{R}_q^r \times \mathbb{R}_p^r \qquad v \mapsto \omega(v, \cdot)$

$n = 2r \qquad V \to V^*$

$$\omega = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

pseudo inner product

$$\omega\left(\frac{dx}{dt}, \cdot\right) = \nabla H$$

$dH \qquad d = \text{ext. der.}$

class. mech. is a Hamiltonian system

with $p_k = m_k v_k$,

$$H(q,p) = \sum \frac{p_k^2}{2m_k} - \sum_{j<k} \text{const}_{jk} \frac{1}{|q_j - q_k|}$$

<u>Systems of particles</u> = subset S of $\{1 \dots N\}$

time-dep., <u>Ex</u> $\Lambda \subseteq \mathbb{R}^3$, $S = \{k \in \{1 \dots N\} : q_k \in \Lambda\}$

<u>Def</u> S <u>closed</u> or <u>isolated</u> iff behaves as if alone in the universe

effective eq. of motion

special case: $H = \sum \frac{p_k^2}{2m_k} + V(q_1 \cdots q_N)$ potential fct

$$V(q_1 \cdots q_N) = \sum_{k=1}^{N} V_1(q_k) + \sum_{j<k} \underbrace{V_2(q_j, q_k)}_{\text{pair potential}} .$$

external potential

$V_1(x,y,z) = -mgz$

Ex Coulomb inter.

$$V_2(q, q') = \frac{\text{const}}{|q - q'|}$$

Flow

$\Omega \subseteq \Gamma$, where sols exist globally.

solution map $T : \mathbb{R} \times \Omega \to \Omega$

$T(t,x) = T^t(x) = T^t x$ s.t. $T^t x(0) = x(t)$

flow map. If F is time indep.

then

$$T^0 = id, \qquad T^t T^s = T^{s+t}$$

$\Big\}$ one-parameter group, $t \mapsto T^t$

$(T^t)^{-1} = T^{-t}$ b/c $T^t T^{-t} = T^{t-t} = T^0 = id$.