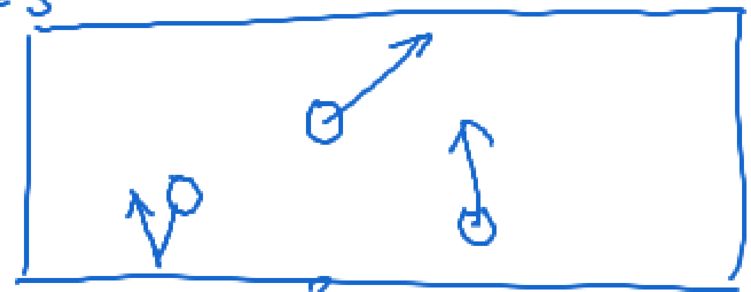
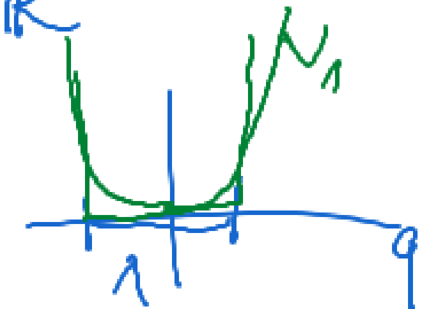
Billiards = Hard Sphere gas



$\Lambda \subset \mathbb{R}^3$



radius $= a$

first description:

$$V_1(q) = \begin{cases} 0 & \text{if } q \in \Lambda \\ \infty & \text{if } q \notin \Lambda \end{cases}$$

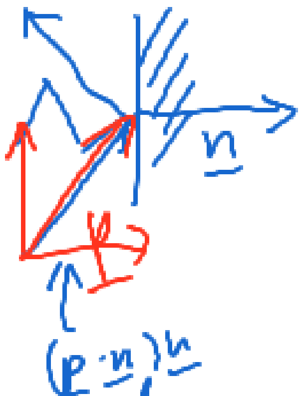
$$V_2(q_1, q_2) = \begin{cases} 0 & \text{if } |q_1 - q_2| > 2a \\ \infty & \text{if } |q_1 - q_2| \leq 2a \end{cases}$$

second description

$$\Gamma = (\Lambda \times \mathbb{R}^3)^N \setminus \{ |q_i - q_j| < 2a \text{ for some } i \neq j \}$$

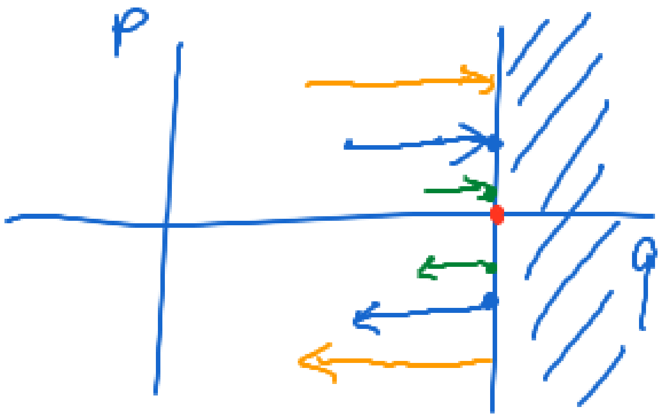
Collision with the wall: $\partial\Lambda$ piecewise C^1

$\underline{n}$ = unit normal vector outward



$$\underline{p} \rightsquigarrow \underline{p}' = \underline{p} - 2(\underline{p}\cdot\underline{n})\underline{n}.$$

specular reflection



Collision between balls:

whenever $|q_1 - q_2| = 2a$, momenta change

$$(\underline{p}_1, \underline{p}_2) \rightsquigarrow (\underline{p}_1', \underline{p}_2')$$

Conservation of energy, momentum & ang. momentum

Assume balls don't spin

2 possibilities: 1) $\underline{p}_1' = \underline{p}_1$, $\underline{p}_2' = \underline{p}_2$

2) $\underline{p}_1' = \underline{p}_1 - \left[(\underline{p}_1 - \underline{p}_2) \cdot \underline{\omega} \right] \underline{\omega}$

$\underline{p}_2' = \underline{p}_2 + \left[(\underline{p}_1 - \underline{p}_2) \cdot \underline{\omega} \right] \underline{\omega}$

$\underline{\omega} := (q_2 - q_1)/2a$

provided $(\underline{p}_1 - \underline{p}_2) \cdot \underline{\omega} > 0$ moving towards each other

Thm (R.K. Alexander 1976,
L.N. Vaserstein 1979)

One can delete a set Γ_b of volume 0 from the hard sphere phase space Γ_0 such that the rules above define the motion uniquely for all $t \in \mathbb{R}$ for all $x(0) \in \Gamma_0 \setminus \Gamma_b =: \Gamma$.

Review of Probability and Measure

Kolmogorov 1933: probability is a normalized measure on a σ-algebra over the history space (= sample space) Ω.

Def A σ-algebra over a set Ω = family $\mathcal{F}$ of subsets s.t. 1) $\emptyset \in \mathcal{F}$, 2) $\forall A \in \mathcal{F}$, $A^c = \Omega \setminus A \in \mathcal{F}$, 3) If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

$(\Omega, \mathcal{F})$ is called a *measurable space*, $A \subseteq \Omega$ is *measurable* iff $A \in \mathcal{F}$.

Def A **measure** μ on a measurable space $(\Omega, \mathcal{F})$ is a mapping $\mu : \mathcal{F} \to [0, \infty]$ (set function) that is σ-additive, i.e., for any $A_1, A_2, \ldots \in \mathcal{F}$ with $A_j \cap A_k = \emptyset$ $\forall j \neq k$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

$(\Omega, \mathcal{F}, \mu)$ is called a **measure space**.

$\Rightarrow \mu(\emptyset) = 0$

$\Rightarrow \mu(A) \leq \mu(\Omega)$. μ is **finite** iff $\mu(\Omega) < \infty$, normalized iff $\mu(\Omega) = 1$

σ-finite iff $\exists A_1, A_2, \ldots : \mu(A_n) < \infty$

and $\bigcup_{n=1}^{\infty} A_n = \Omega$.

surprising: 1) not every $A \subseteq \Omega$ is measurable.

2) not enough that μ is additive.

3) not uncountably-additive

On $\mathbb{R}^d$, Borel σ-algebra (generated by the family of open sets).

Lebesgue σ-algebra

volume = Lebesgue measure on Borel σ-alg
or on Lebesgue σ-alg.

Thm $\exists_1$ measure λ on the Borel σ-alg of $\mathbb{R}^d$ that is translation invariant,

$$\lambda(A) = \lambda(\underbrace{A + \underline{a}}_{\{\underline{x}+\underline{a} \,:\, \underline{x} \in A\}}) \qquad \forall \underline{a} \in \mathbb{R}^d$$

and $\lambda([0,1)^d) = 1$, $\qquad \lambda =$ Lebesgue measure

A <u>null set</u> $\Leftrightarrow \lambda(A) = 0$.

λ inv. under $O(d)$: $\lambda(RA) = \lambda(A) \quad \forall R \in O(d)$.

Prop There is no measure μ on the power set of $[0,1)$ that is translation inv. in the periodic sense $A+a = \{x+a \mod 1 : x \in A\}$ and $\mu([0,1)) = 1$.

Ex $0.8 + 0.5 \mod 1$
$= 0.3$

$x \equiv y \mod 1$
$:\Leftrightarrow x-y \in \mathbb{Z}$

Pf Consider $a \in \mathbb{Q}$.

$x \sim y :\Leftrightarrow x-y \in \mathbb{Q}$.
equiv. rel.

Choose 1 from each orbit, form A. (axiom of choice)

$\forall x \in [0,1)\ \exists_1 (y,a) \in A \times \mathbb{Q} : x = y+a$

$$[0,1) = \bigcup_{a \in \mathbb{Q}} A + a$$

$$1 = \mu([0,1)) = \mu\Big(\bigcup_a A + a\Big) =$$

$$= \sum_{a \in \mathbb{Q}} \underbrace{\mu(A+a)}_{\mu(A)} \quad ↯ \qquad \square$$

Also Banach-Tarski thm

uncountable additivity

$$\lambda(\{x\}) = 0$$

$$\lambda([a,b]) = b-a$$

$$\sum_{x\in[0,1]} \lambda(\{x\}) = 0 \neq 1 = \lambda([0,1]) = \lambda\Big(\bigcup_{x\in[0,1]}\{x\}\Big)$$

finite additivity: $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$

$\bigcap_{n\in\mathbb{N}} A_n = \emptyset$ but $\lim_{n\to\infty} \mu(A_n) > 0$.

Cournot's Principle

"the only connection between μ and reality" (Paul Lévy 1919)

If $\mu(A)$ is very small, then one can be practically certain that A will not occur.

for any event A singled out in advance,