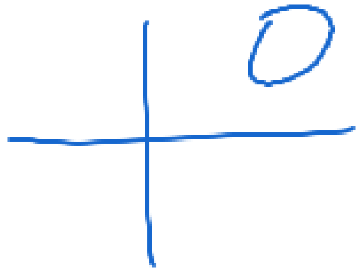
$\cancel{\mathcal{F}}$ $\mathbb{C}$  $\int d^2z\, f(z)$

$$\mu : \mathcal{F} \to \mathbb{C}, \qquad \mu(B) = \underbrace{\lambda(B)}_{\text{signed measure}} + i\,\underbrace{\nu(B)}_{\text{signed measure}}$$

$$\lambda(B) = \lambda^+(B) - \lambda^-(B)$$

Integration

~~Riemann (1854)~~ Lebesgue (1902)

Def $f : X \to Y$ <u>measurable</u> $\Longleftrightarrow$

$(X, \mathcal{F}_X)$, $(Y, \mathcal{F}_Y)$ $\qquad \forall A \in \mathcal{F}_Y : \underbrace{f^{-1}(A)}_{} \in \mathcal{F}_X$

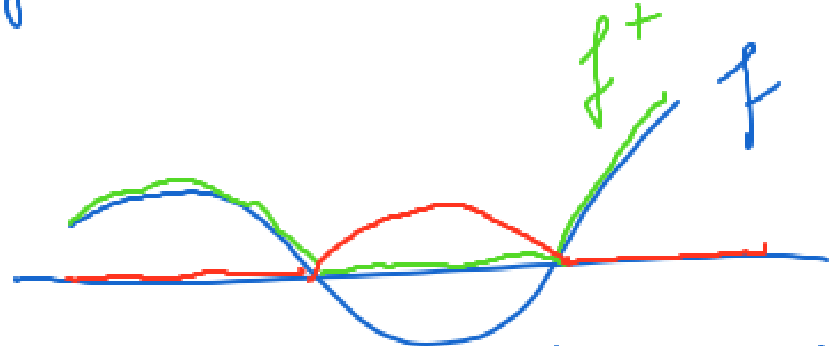
$\{x \in X : f(x) \in A\}$.

Prop $f : X \to [0, \infty)$ measurable, μ meas. on X

then $\int_X \mu(dx) f(x)$ is well defined, $\in [0, \infty]$.

$$f^+(x) = \max\{0, f(x)\}$$

$$f^-(x) = \max\{0, -f(x)\}$$



$$f = f^+ - f^-$$

Def $f : X \to [0, \infty)$, integrable $\iff \int_X \mu(dx) f(x) < \infty$

Def $f : X \to \mathbb{R}$ integrable $\iff f^+, f^-$ are int'ble

$$\int f := \int f^+ - \int f^-$$

__Ex__ prob. density

ν meas. on $(\Omega, \mathcal{F})$, has __density fct__ relative to μ meas. on Ω

$\rho : \Omega \to [0,\infty)$

$\Longleftrightarrow$ ρ meas.ble, $\nu(A) = \int_A \mu(d\omega)\, \rho(\omega)$

$\rho =$ Radon-Nikodym derivative $= \dfrac{d\nu}{d\mu}$

__Def__ ν is __absolutely continuous rel. to μ__ $\Longleftrightarrow$ every ~~ν-nulls~~ μ-null set is a ν-null set.

__Thm__ (Radon-Nikodym) If ν, μ σ-finite, ν abs. cont. μ, then $\exists \rho$, unique up to changes on μ-null sets.

Ex expectation value

prob. space $(\Omega, \mathcal{F}, \mathbb{P})$, random variable = r.v.

$$X : \Omega \longrightarrow M$$

vector space

$$\mathbb{E}X = \langle X \rangle = \int_\Omega \mathbb{P}(d\omega)\, X(\omega)$$

Rem linear

variance: $M = \mathbb{R}$, $\operatorname{Var} X := \mathbb{E}\left[(X - \mathbb{E}X)^2\right]$

$$= \mathbb{E}(X^2) - (\mathbb{E}X)^2 = \sigma_X^2$$

covariance: X, Y, $\operatorname{Cov}(X,Y) := \mathbb{E}\left[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\right]$

Rem bilinear

$$= \mathbb{E}(XY) - (\mathbb{E}X)(\mathbb{E}Y).$$

$\mathrm{Var}(X) = \mathrm{Cov}(X, X)$

Ex If X assumes only 0 and 1, then

$$\mathbb{E}X = 1 \cdot \underbrace{\mathbb{P}(X=1)}_{=: p} + 0 \cdot \underbrace{\mathbb{P}(X=0)}_{1-p=q} = p$$

and $\mathrm{Var}\,X = pq$

Pf $X^2 = X$, so $\mathrm{Var}\,X = \mathbb{E}(X^2) - (\mathbb{E}X)^2$

$$= p - p^2 = p(1-p). \square$$

Product measures

$\mathcal{M}_1 \times \cdots \times \mathcal{M}_n$

$\mu_1 \times \cdots \times \mu_n$ is

defined uniquely by

$$\underbrace{\mu_1 \times \cdots \times \mu_n}_{=:\mu \text{ prod. measure}} (A_1 \times \cdots \times A_n) = \mu_1(A_1) \cdots \mu_n(A_n).$$

whenever $\mu_j(A_j) < \infty \ \forall j$.

μ always exists, unique if μ_j σ-finite $\forall j$.

Countable product: $\mu_1, \mu_2, \ldots,$ $\mu_1 \times \mu_2 \times \cdots$ exists if μ_j normalized, but not in general.

<u>marginal distribution</u> of r.v. X ($\mathcal{M}$-valued)

is $\mu(A) = \mathbb{P}(X \in A)$, $\quad X \sim \mu$

<u>independent</u> $X_1 \dots X_n :\Longleftrightarrow$ joint distr. is the prod. of their marginal distributions.

$X_1, X_2, \dots$ in<u>dependent</u> $:\Longleftrightarrow$ every finite subfamily is indep.

Conditional probability, $A, B \in \mathcal{F}$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \quad \text{provided } \mathbb{P}(B) \neq 0.$$

r.v. X discrete, conditional distr. of Y, given $X = x$

$$\mathbb{P}(Y \in A | X = x) = \frac{\mathbb{P}(Y \in A, X = x)}{\mathbb{P}(X = x)} \longleftarrow \text{if } \neq 0.$$

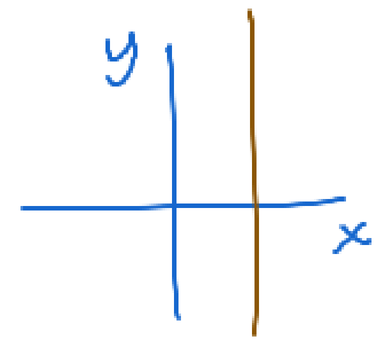
X, Y indep $\Longleftrightarrow$ $\mathbb{P}(Y \in A | X = x) = \mathbb{P}(Y \in A | X = x') \ \forall x, x'$

X cont. real r.v. ; $\mathbb{P}(X=x)=0$.

But if X, Y jointly cont., i.e., $\exists\, \rho_{XY}(x,y)$

then $$\rho(Y=y \mid X=x) = \frac{\rho_{XY}(x,y)}{\int dy'\, \rho_{XY}(x,y')}$$

cond. density



Again, X, Y indep. $\Longleftrightarrow$ $\rho(Y=y \mid X=x)$ indep of x.

conditional expectation $\mathbb{E}(Y \mid X=x)$

Rem If X, Y indep, real-valued, then $\mathbb{E}(XY) = (\mathbb{E}X)(\mathbb{E}Y)$, so $\mathrm{Cov}(X,Y)=0$

Conseq $\underline{\mathrm{Var}(X+Y)} = \mathrm{Cov}(X+Y, X+Y)$

$$= \mathrm{Cov}(X,X) + \mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X) + \mathrm{Cov}(Y,Y)$$

$$= \mathrm{Var}(X) + 2\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y)$$

$$\overset{\text{indep.}}{=} \underline{\mathrm{Var}(X) + \mathrm{Var}(Y)}$$

Gaussian distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$\mathbb{E}X = \mu$, $\mathrm{Var}(X) = \sigma^2$, $X \sim \mathcal{N}(\mu, \sigma^2)$

Def A gaussian distr. in $\mathbb{R}^d$ is a cont. distr. with density $p(\underline{x}) = \exp(P(\underline{x}))$, where

$\underline{x} = (x_1 \dots x_d)$ $\quad P$ is a real poly of degree 2.

$$P(x_1 \dots x_d) = \sum_{k=1}^{d} \alpha_k x_k^2 + \sum_{k=2}^{d} \sum_{j=1}^{k-1} \beta_{jk} x_j x_k + \sum_{k=1}^{d} \gamma_k x_k + \delta$$

or equivalently

$$P(x_1 \dots x_d) = \sum_{j,k=1}^{d} a_{jk} x_j x_k + \sum_{k=1}^{d} \overset{\gamma_k}{\cancel{\beta_k}} x_k + \delta$$

$$A = (a_{jk})$$

$$a_{jj} = \alpha_j, \quad \underset{j<k}{a_{jk} = \frac{\beta_{jk}}{2}}, \quad \underset{j>k}{a_{jk} = \frac{\beta_{kj}}{2}} \quad \Rightarrow \quad A = A^T$$

symm. transpose

Symm. part $\frac{1}{2}(A + A^T)$

$$P(\underline{x}) = \underline{x}^T A \underline{x} + \underline{\gamma}^T \underline{x} + \delta$$

$$\int e^{P(\underline{x})} d\underline{x} < \infty \quad \Longrightarrow \quad \underline{x}^T A \underline{x} < 0 \quad \text{for } \underline{x} \neq \underline{0}$$

i.e. A <u>neg. def.</u>

$$P(\underline{x}) = (\underline{x}-\underline{\mu})^T A (\underline{x}-\underline{\mu}) + \delta' \quad \text{with} \quad \underline{\mu} = -\frac{1}{2} A^{-1} \underline{\gamma}$$

$\Rightarrow$ Gaussian distr. = translates of

$$\rho(\underline{x}) = \mathcal{N} e^{\underline{x}^T A \underline{x}}$$

<u>spectral thm for symm. matrices</u>

$\forall$ symm. A $\exists$ ONB of $\mathbb{R}^d$ that diagonalizes A.

That is $\quad \underline{\bar{x}} = R\underline{x}, \quad \bar{\rho}(\underline{\bar{x}}) = \mathcal{N} e^{\underline{\bar{x}}^T \bar{A} \underline{\bar{x}}}$

$$= \mathcal{N} e^{\sum_{k=1}^{d} a_k \bar{x}_k^2}$$

Thus, $\bar{\rho}(\underline{x}) = \prod_{k=1}^{d} \frac{1}{\sqrt{2\pi}\,\sigma_k} e^{-\frac{x_k^2}{2\sigma_k^2}}$

$$a_k = -\frac{1}{2\sigma_k^2}$$

i.e. $\underline{\bar{X}} = (\bar{X}_1 \dots \bar{X}_d)$ indep. 1d gaussian

BTW, $\mathbb{E}\underline{\bar{X}} = \underline{0}$, so $\mathbb{E}\underline{X} = \mathbb{E}(R^{-1}\underline{\bar{X}}) = R^{-1}\underbrace{\mathbb{E}(\underline{\bar{X}})}_{=\underline{0}}$

so after translation $\mathbb{E}\underline{X} = \underline{\mu}$.

level surface of $\rho(\underline{x})$

<u>Covariance matrix</u> $C = (C_{ij})$

$C_{ij} = \mathrm{Cov}(X_i, X_j)$ symm. $d \times d$.

For Gaussian, $C = -\frac{1}{2} A^{-1}$. pos. def.

$C_{ii} = \sigma_i^2 = -\frac{1}{2a_i}$,

$\underline{\mathcal{N}^d(\underline{\mu}, C)}$

$\forall \underline{\mu} \in \mathbb{R}^d$

$\forall$ pos. def. symm C.