Reasons behind the Maxwellian:

Maxw. as the marginal distr.

as the ~~ex~~ typical emp. distr.

ideal gas = non-interacting

$\Gamma = \Lambda^N \times \mathbb{R}^{3N}$ phase space,

$\Lambda \subseteq \mathbb{R}^3$ $\qquad H = \sum_{k=1}^{N} \frac{p_k^2}{2m} = \frac{1}{2m} \|p\|_2^2$

$\Gamma_E = \{(q,p) \in \Gamma : H(q,p) = E\}$

Here, $\Gamma_E = \Lambda^N \times S^{d-1}_{\sqrt{2mE}}$

$\underbrace{\text{purely random}}_{\text{uniformly distr.}}$ point on $S_R^{d-1}$

$\mathcal{U}_R^{d-1}$

<u>Fact</u> If $d \gg 1$, then 3d marginal of $\mathcal{U}_R^{d-1}$ is close to $\mathcal{N}^3(\underline{0}, R^2 d^{-1} \underline{I})$. That is,

<u>Thm</u> Let $\sigma > 0$, $k \in \mathbb{N}$, $\rho_{\text{Gauss}} : \mathbb{R}^k \to [0,\infty)$ the density of $\mathcal{N}^k(\underline{0}, \sigma^2 I)$. Consider $d > k$, $(X_1 \dots X_d) \sim \mathcal{U}_R^{d-1}$ with $R = \sigma\sqrt{d}$. The marginal

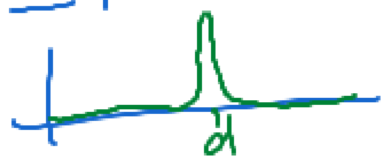
$\mu_{k,d}$ of $X_1 \dots X_k$ is abs. cont. in $\mathbb{R}^k$
with density $\rho_{k,d}$, and

$$\|\rho_{k,d} - \rho_{Gauss}\|_{L^1(\mathbb{R}^k)} \xrightarrow{d\to\infty} 0.$$

Plausible: $Z_1 \dots Z_d$ i.i.d. $\mathcal{N}^1(0,1)$

$\Rightarrow \underline{Z} = (Z_1 \dots Z_d) \sim \mathcal{N}^d(\underline{0}, I)$

$\underline{X} = \dfrac{R\underline{Z}}{|\underline{Z}|} \sim u_R^{d-1}$, $d \gg 1$, $|\underline{Z}|^2 = Z_1^2 + \dots + Z_d^2 \approx d$.



Pf Step 1:

$$\rho_{k,d,R}(\underline{x}) = \frac{A_{d-k}}{A_d} \frac{1}{R^2} 1_{\underline{x}^2 \leq R^2} \left(R^2 - \underline{x}^2\right)^{\frac{d-k}{2}-1}$$

$\underline{x} \in \mathbb{R}^k$

Lemma 1 ("uniform conv. $\Longrightarrow$ $L^1$ conv.")

If $\rho_d$ and $\rho$ are normalized densities on $\mathbb{R}^k$ and $\rho_d \xrightarrow{d\to\infty} \rho$ uniformly, then

$$\| \rho_d - \rho \|_{L^1(\mathbb{R}^k)} \longrightarrow 0 \quad \text{as } d \to \infty.$$

Pf of Lemma 1: Let $0 < \varepsilon < 1$.

Choose ball $B$ around $\underline{0}$ so that $\int_B \rho > 1 - \varepsilon$.

By unif. conv., $\exists d_0 \; \forall d > d_0 \; \forall \underline{x}: \left| \rho_d(\underline{x}) - \rho(\underline{x}) \right| < \frac{\varepsilon}{\mathrm{vol}(B)}$.

$$\Rightarrow \int_B \underline{|\rho_d - \rho|} < \underline{\varepsilon}.$$

Outside: $\int_B (-\rho_d + \rho) < \varepsilon \Rightarrow \int_B \rho_d > \int_B \rho - \varepsilon > 1 - 2\varepsilon$

$$\Rightarrow \int_{B^c} \rho_d < 2\varepsilon \Rightarrow \int_{B^c} \underline{|\rho_d - \rho|} \leq \int_{B^c} (\rho_d + \rho) < 2\varepsilon + \underline{\varepsilon}$$

$$\Rightarrow \int_{\mathbb{R}^d} |\rho_d - \rho| < 4\varepsilon. \quad \square$$

## Lemma 2

$$\tilde{\rho}_{k,d}(\underline{x}) = 1_{\underline{x}^2 \leq \sigma^2 d} \left(1 - \frac{\underline{x}^2}{\sigma^2 d}\right)^{-\frac{k}{2}-1} \left(1 - \frac{\underline{x}^2/2\sigma^2}{d/2}\right)^{\frac{d}{2}}$$

$$\xrightarrow{d \to \infty} e^{-\underline{x}^2/2\sigma^2} = \tilde{\rho}(\underline{x}) \text{ uniformly in } \mathbb{R}^k$$

Pf

$$\left(1 + \frac{x}{n}\right)^n \xrightarrow{n \to \infty} e^x$$ uniformly on every compact set in $\mathbb{R}$.

Let $0<\varepsilon<1$, choose $r>0$ so that $e^{-r^2/2\sigma^2} < \frac{\varepsilon}{3}$.

Then for $|\underline{x}| \geq r$, $\tilde{\rho}(\underline{x}) < \frac{\varepsilon}{3}$ and

$$\tilde{\rho}_{k,d}(\underline{x}) = \tilde{\rho}_{k,d}(|\underline{x}|, 0 \dots 0) \leq \tilde{\rho}_{k,d}(r, 0 \dots 0) \leq \tilde{\rho}(r, 0 \dots 0) + \frac{\varepsilon}{3}$$

$$\Rightarrow |\tilde{\rho}_{k,d}(\underline{x}) - \tilde{\rho}(\underline{x})| \leq \tilde{\rho}_{k,d}(\underline{x}) + \tilde{\rho}(\underline{x}) < \varepsilon$$

For $|\underline{x}| \leq r$: uniform conv.

$\Rightarrow$ uniform conv. on $\mathbb{R}^k$. $\square$

Last step: $\mathcal{N}_{k,d} \longrightarrow \mathcal{N} \quad d \to \infty$

$$\rho_{k,d}(\underline{x}) = \mathcal{N}_{k,d}\, \tilde{\rho}_{k,d}(\underline{x})$$

Lemma 1

$\Longrightarrow$ Thm $\square$

Upshot: For a purely random phase point, the distr $\underline{v}_1$ is (approx.) Maxwellian.

That is, fix $\bar{e}$, set $E = N\bar{e}$, consider

$$\underline{X} \sim \mathcal{U}^{3N-1}_{\sqrt{2mE}} \Rightarrow (X_1, X_2, X_3) \xrightarrow{N\to\infty} \mathcal{N}^3\left(\underline{0}, \frac{2}{3} m\bar{e}\,\underline{I}\right)$$

Maxw.

## Maxw. as the typical emp. distr.

**Claim:** For most $x \in S^{3N-1}_{\sqrt{2mE}}$ with $N \gg 1$, the emp. distr. of momenta is close to the Maxw.

**Thm** Let $\bar{\varepsilon} > 0$, $\mu = \mathcal{N}^3\left(\underline{0}, \frac{2}{3} m \bar{\varepsilon} \underline{\mathbb{1}}\right)$ Maxw. for momenta, $\mathcal{A} = \{A_1 \dots A_r\}$ partition of $\mathbb{R}^3$. For any $N$, let $E = N\bar{\varepsilon}$, $(\underline{X}_1 \dots \underline{X}_N) \sim u_R^{3N-1}$ with $R = \sqrt{2mE}$. Let $(F_1 \dots F_r)$, $F_i = \frac{1}{N} \# \{k \leq N : \underline{X}_k \in A_i\}$, coarse-grained emp. distr.

$$\forall \varepsilon > 0: \quad \mathbb{P}\left(\forall i: |F_i - \mu(A_i)| < \varepsilon\right) \xrightarrow{N \to \infty} 1.$$

Plausible: weakly dep. $\underline{X}_j$ $\Longleftarrow$ Thm about marginal

$$\underline{X}_k = \frac{R\,\underline{Z}_k}{|\underline{Z}|}$$

permutation invariant = "exchangeable"

Pf $\quad P_i := \mathbb{P}(\underline{X}_1 \in A_i)$ $\leftarrow k$

Thm about marginal: $P_i \longrightarrow \mu(A_i)$ as $N \to \infty$.

$$F_i = \frac{1}{N}\sum_{k=1}^{N} \mathbb{1}_{\underline{X}_k \in A_i}, \quad \mathbb{E}F_i = P_i$$

$$\operatorname{Var} F_i = \operatorname{Cov}(F_i, F_i)$$

$$= \frac{1}{N^2} \sum_{j,k=1}^{N} \operatorname{Cov}\left(\mathbb{1}_{X_j \in A_i}, \mathbb{1}_{X_k \in A_i}\right)$$

$$= \frac{1}{N^2} \sum_{j=1}^{N} \underbrace{\operatorname{Var}\left(\mathbb{1}_{X_j \in A_i}\right)}_{P_i(1-P_i)} + 2N^{-2} \sum_{j<k} \operatorname{Cov}\left(\mathbb{1}_{X_j \in A_i}, \mathbb{1}_{X_k \in A_i}\right)$$

$$\leq \frac{P_i(1-P_i)}{N} + \cancel{2N^{-2}} \; \frac{\cancel{N^2}\quad \cancel{N^2 - N}}{\cancel{2}} \left| \operatorname{Cov}\left(\mathbb{1}_{X_1 \in A_i}, \mathbb{1}_{X_2 \in A_i}\right) \right|$$

$$\leq \frac{p_i(1-p_i)}{N} + \Big| \underbrace{\mathbb{E}\left(1_{\underline{X}_1\in A_i} 1_{\underline{X}_2\in A_i}\right)}_{\mathbb{P}(\underline{X}_1\in A_i,\, \underline{X}_2\in A_i)} - \underbrace{\left(\mathbb{E}\, 1_{\underline{X}_1\in A_i}\right)}_{\mathbb{P}(\underline{X}_1\in A_i) = p_i} \overbrace{\left(\mathbb{E}\quad\right)}^{p_i} \Big|$$

$$p_i \xrightarrow{N\to\infty} \mu(A_i)$$

$$\xrightarrow{N\to\infty} 0 \quad \text{b/c}$$

$$p_i^2 \longrightarrow \mu(A_i)^2$$

$$\mathbb{P}(\underline{X}_1\in A_i, \underline{X}_2\in A_i) \xrightarrow{N\to\infty} \cancel{p_i^2}\ \mu(A_i)^2$$

b/c Thm about marginal

Upshot: $\operatorname{Var} F_i \longrightarrow 0$.

Chebyshev ineq: If $|P_i - \mu(A_i)| < \frac{\varepsilon}{2}$

$$\mathbb{P}\Big(\bigcup_i \{|F_i - \mu(A_i)| \geq \varepsilon\}\Big)$$

$$\leq \mathbb{P}\Big(\bigcup_i \{ |F_i - P_i| + |P_i - \mu(A_i)| \geq \varepsilon \}$$

$$\leq \sum_i \mathbb{P}\Big(|F_i - P_i| \geq \frac{\varepsilon}{2}\Big)$$

$$\overset{Ch.}{\leq} \sum_{i=1}^{r} \frac{\operatorname{Var} F_i}{\varepsilon^2/4} \xrightarrow{N \to \infty} 0. \qquad \square$$