Thus <u>Most</u> phase points have on the energy surface the Maxwellian emp. distr.

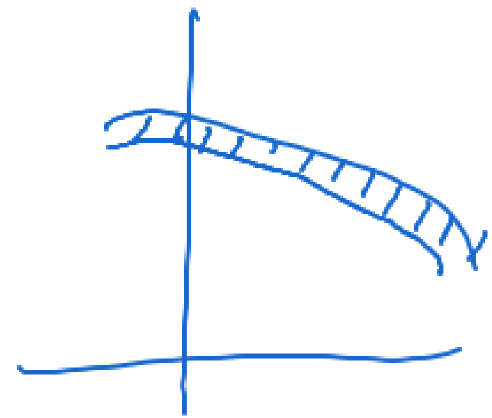
"The Maxwellian is <u>typical</u>."

$$\bar{e} = \frac{3}{2} kT \,, \qquad \bar{e} = \frac{E}{N}$$

Probability Spaces in Classical Mech.

Gibbs (1869 - 1903) : "ensembles"

- micro-canonical ens.
- canonical ens.



$$\Gamma_E = \{ x \in \Gamma : H(x) = E \}$$

$$p_{can}(x) = \frac{1}{Z} e^{-\beta H(x)}, \quad \beta = \frac{1}{kT} \text{ inverse temp.}$$

$Z(\beta)$ = partition fct.

Liouville's thm

Hamiltonian system

$\Gamma \subseteq \mathbb{R}^{2r}$ $\quad H : \Gamma \to \mathbb{R}$ smooth

Ass. Every sol. exists globally.

Suff. cond. $\Gamma = \mathbb{R}^{2r}$, Γ_E compact $\forall E$.

Convey $\quad T^t : \mathbb{R}^{2r} \to \mathbb{R}^{2r}$ bij.

~~Thm~~ Liouville's thm $\quad \forall A \subseteq \mathbb{R}^{2r}$:

$$\mathrm{vol}_{2r}(T^t A) = \mathrm{vol}_{2r}(A).$$

Thm $\frac{dx}{dt} = F(x)$ time indep.

suppose sol.s exist globally.

If $\nabla \cdot F = 0$, then T^t preserves vol.

Pf $T^t(x) = x + F(x)t + O(t^2)$.

Def $g(t) = O(f(t)) :\Longleftrightarrow \frac{g(t)}{f(t)}$ bdd as $t \to 0$.

$v(t) = \mathrm{vol}(T^t A)$

claim

$$\left.\frac{dv}{dt}\right|_{t=0} = \int_A d^d x \; \nabla \cdot F$$

trasformation formula

$$v(t) = \int_{T^t A} d^d y \overset{y = T^t x}{=} \int_A d^d x \left| \det \frac{\partial T^t}{\partial x} \right|$$

Jacobian matrix $J = \dfrac{\partial T^t}{\partial x}$ means $J_{ij} = \dfrac{\partial T^t_i}{\partial x_j}$

$$\frac{\partial T^t}{\partial x} = I + \frac{\partial F}{\partial x} t + O(t^2)$$

$$\det(I + Mt) = 1 + t \operatorname{tr} M + O(t^2) \quad \text{as } t \to 0$$

$$v(t) = \int_A d^d x \, \underbrace{\det \frac{\partial T^t}{\partial x}}_{1 + t \, \underbrace{\mathrm{tr} \frac{\partial F}{\partial x}}_{\sum_i \frac{\partial F_i}{\partial x_i}} + O(t^2) \; = \; 1 + t \nabla \cdot F + O(t^2)} = v(0) + t \int_A d^d x \, \underbrace{\nabla \cdot F}_{0} + O(t^2)$$

$$\Longrightarrow \quad \left. \frac{dv}{dt} \right|_{t=0} = 0 \qquad \square$$

Pf of Liouville's thm: $F = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}$

$$\Rightarrow \nabla \cdot F = \sum_{i=1}^{r} \left(\frac{\partial}{\partial q_i} \frac{\partial H}{\partial p_i} + \frac{\partial}{\partial p_i} \left(-\frac{\partial H}{\partial q_i} \right) \right) = 0$$

$\square$

<u>Cor</u> $\frac{dx}{dt} = F(x)$, $x(0) \sim \rho_0\, dx$. Then $x(t) \sim \rho_t\, dx$

and

$$\frac{\partial \rho_t}{\partial t} = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\rho_t(x)\ F_i(x) \right)$$

Li. thm $\Leftrightarrow$ Lebesgue meas. is invariant

$$\Rightarrow \quad \mu_{can}(A) = \int_A dx\, \rho_{can}(x) \quad \text{is invariant.}$$

i.e. $\mu_{can}(T^t A)$

$= \mu_{can}(A).$

Microcanonical Ensemble

Γ_E , $\Gamma_{E,\Delta E} = \{ x \in \Gamma : E - \Delta E < H(x) \leq E \}$

[Reason: e.g., $H(q,p) = \frac{|p|^2}{2m}$, oranges are mostly peel.]

$$\mu_{mc}(A) = \frac{\mathrm{vol}(A \cap \Gamma_{E,\Delta E})}{\mathrm{vol}(\Gamma_{E,\Delta E})}$$

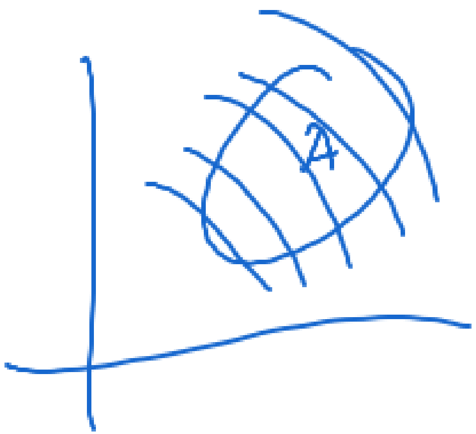
$$\rho_{mc}(x) = \frac{1}{\mathrm{vol}(\Gamma_{E,\Delta E})} 1_{\Gamma_{E,\Delta E}}(x)$$

Now consider $\Delta E \to 0$,

$$\rho_{mc}(x) = \mathcal{N}\,\delta(E - H(x))$$

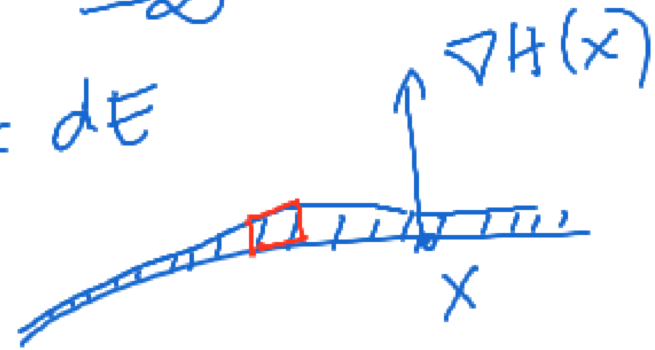
$$\mu_E(A) = \frac{\lambda_E(A)}{\lambda_E(\Gamma_E)}, \qquad \lambda_E \text{ "cond. vol"}$$

def'd by $\mathrm{vol}(A) = \int_{-\infty}^{\infty} dE\, \lambda_E(A \cap \Gamma_E)$



(thickness) $|\nabla H(x)| = dE$

$\Rightarrow$ thickness $= \dfrac{dE}{|\nabla H(x)|}$



$$\frac{d\,\lambda_E}{d\,\mathrm{area}_{T_E}} = \frac{1}{|\nabla H(x)|}$$

Measure-Preserving Dynamical Systems

Def MPDS $(\Omega, \mathcal{F}, \mathbb{P}, T)$

a) in cont. time: $T : \mathbb{R} \times \Omega \to \Omega$ with

$T^0 = id$, $T^s T^t = T^{s+t}$, $\mathbb{P}(T^t A) = \mathbb{P}(A)$

b) in discrete time: $T : \cancel{\mathbb{Z} \times \Omega \to} \, \Omega$

$T : \Omega \to \Omega$ biji

T^k, $\mathbb{P}(TA) = \mathbb{P}(A) \quad \forall A \in \mathcal{F}.$

Poincaré Recurrence Thm given MPDS in either cont. or discrete time. Let $A \subseteq \Omega$ with $\mathbb{P}(A) > 0$. Then for almost every $\omega \in A$, there exist arbitrarily large t such that $T^t \omega \in A$.