MPDS $(\Omega, \mathcal{F}, \mathbb{P}, T)$

<u>Poincaré recurrence thm</u> $A \subseteq \Omega$, $\mathbb{P}(A) > 0$

Then for almost every $\omega \in A$ there exist arb. large times t s.t. $T^t \omega \in A$.

<u>Pf</u> Wlog discrete.

$$B := \{\omega \in A : T^n \omega \notin A \ \forall n \in \mathbb{N}\}$$

$\Rightarrow$ if $\omega \in B$ then $T^n \omega \notin B$.

$$\left.\begin{array}{l} B \cap T^{-n}(B) = \emptyset \\ T^{-k}(B) \cap T^{-n-k}(B) = \emptyset \end{array}\right\}$$

Thus, $B, T^{-1}(B), T^{-2}(B), \ldots$

are mutually disj., have equal $\mathbb{P}$

$\Rightarrow \mathbb{P}(B) = 0.$

(ii) (iv) (v) (vi) <u>arb. large</u>: If $\omega \in A$ returns finitely many times, $T^k\omega \notin A$ $\forall k \geq n$

then $T^n \omega \in B \iff \omega \in \underbrace{T^{-n}}_{\mathbb{P}=0}(B)$

$\{\omega \in A :$ return finitely often$\}$

$$\subseteq \underbrace{\bigcup_{n=1}^{\infty} T^{-n}(R)}_{\mathbb{P}=0}$$

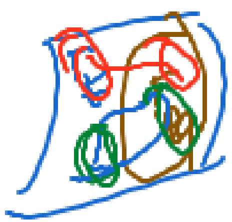
$\square$

Goal: derive thermod. from mech.

Zermelo's paradox:

1) 0th law of thermod.:

every closed system sooner or later reaches th. eq. and remains there.



2) 2nd law of thermod:

~~...~~ entropy increases

3) 

Boltzmann's answer: a) no contradiction

b) t is exorbitantly large.

Thermal equilibrium is not forever, just for 10^N years.

Loschmidt's paradox : time reversal

$U: \Gamma \to \Gamma, \quad U(q,p) = (q,-p)$

$\Gamma_E \to \Gamma_E$, preserves vol.

Identical particles = Permutation symmetry

$$H(q,p) = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N} V_1(q_i) + \frac{1}{2}\sum_{i\neq j} V_2(q_i - q_j)$$

permutation inv.

ordered phase point.

$$x = (x_1, \dots, x_N), \quad x' = (x_{\sigma(1)}, \dots, x_{\sigma(N)})$$

$\sigma \in S_N$

$\{x_1 \dots x_N\}$ unordered phase point

Γ_1 = 1-particle phase space, ${}^N\Gamma_1 := \{X \subset \Gamma_1 : \# X = N\}$

$$\pi(x_1 \dots x_N) = \{x_1 \dots x_N\}$$

forgetful mapping = unordering mapping

$$\pi : \underbrace{\Gamma_1^{N\neq}}_{\{(x_1 \dots x_N) \in \Gamma_1^N \,:\, x_i \neq x_j \ \forall i \neq j\}} \longrightarrow {}^N\Gamma_1$$

many-to-one

N! - to - one

$${}^N\Gamma_1 = \Gamma_1^{N\neq} / S_N$$

motion: $t \mapsto x(t) \in \Gamma_1^N$ sol.

$t \mapsto \sigma(x(t))$ also sol.

F. Conseq: $\pi(x(t))$ dep. only

on $\pi(x(0)) \in {}^N\Gamma_1$

volume in Γ_1^N

$B \subseteq {}^N\Gamma_1$, $\mathrm{vol}(\pi^{-1}(B)) = \mathrm{vol}(\pi^{-1}(B_t))$

overcounting, defines $\mathrm{vol}_{N\Gamma_1}$.

The Canonical Ensemble

obtain can. ens. for system $\mathcal{S}$
as a marginal distr. of micro-can. ens.
of $\mathcal{S} \cup \mathbb{B}$, $\mathbb{B}$ = heat bath = ideal gas
with $N_{\mathbb{B}} \gg 1$.

Ass interaction between $\mathcal{S}$ and $\mathbb{B}$ is
negligible.

Marginal thm: Let $T>0$, $\beta = \frac{1}{kT}$,

$H_S : \Gamma_S \to \mathbb{R}$ such that

$$Z = \int_{\Gamma_S} dx_S \, e^{-\beta H_S(x_S)} < \infty .$$

$\rho_{can}(x_S) = \frac{1}{Z} e^{-\beta H_S(x_S)}$. For $N \in \mathbb{N}$, $\Gamma_B = (\Lambda \times \mathbb{R}^3)^N$

$E = \frac{3}{2} N k T$, $H_{S \cup B}(x_S, x_B) = H_S(x_S) + \frac{p_B^2}{2m}$

Then the marginal of μ_E has density $\rho_{S,N}$

and

$$\| \rho_{S,N} - \rho_{can} \|_{L^1(\Gamma_S)} \xrightarrow{N \to \infty} 0 .$$

Starting from μ_{can},

$$\rho_{\mathcal{S}\cup\mathcal{B}}(x_{\mathcal{S}}, x_{\mathcal{B}}) = \frac{1}{Z} \underbrace{e^{-\beta(H_{\mathcal{S}}(x_{\mathcal{S}}) + H_{\mathcal{B}}(x_{\mathcal{B}}))}}_{e^{-\beta H_{\mathcal{S}}(x_{\mathcal{S}})}\, e^{-\beta H_{\mathcal{B}}(x_{\mathcal{B}})}}$$

Maxwell-Boltzmann Distribution

$\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n \cup \mathcal{B}$, non-interacting

$$\mu_E \rightsquigarrow \quad \rho_1(x_1) = \frac{1}{Z_1} e^{-\beta H_1(x_1)}$$

as marginal or as typical emp.

Ex $H_1(x) = \frac{p^2}{2m} + V_1(q)$

e.g. $V_1(q) = mg\,q_3$.

$$\Rightarrow \quad \rho_1(q,p) = \frac{1}{Z} \underbrace{e^{-mgq_3/kT}}_{\text{barometric formula}} \underbrace{e^{-\frac{p^2}{2mkT}}}_{\text{Maxwellian}}$$

Conseq 1) temp is const.

2) pressure $p(q)$ is not const.