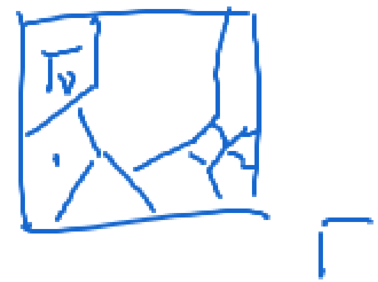
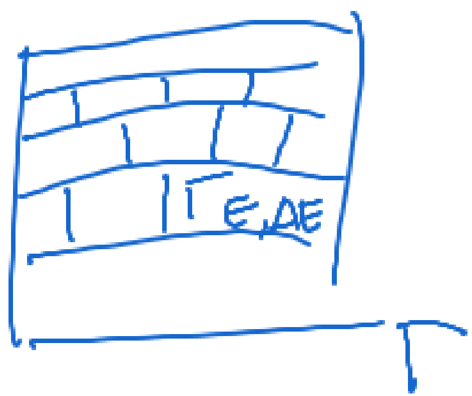
Boltzmann entropy

$$S(x) = k \log \mathrm{vol}\, \Gamma_\nu , \qquad \nu \text{ s.t. } x \in \Gamma_\nu$$



$$S : \Gamma \to \mathbb{R}$$



$$\cancel{S(\Gamma_{eq})}\ \cancel{\approx}\ S(eq) \approx k \log \mathrm{vol}(\Gamma_{E,\Delta E}$$

$$\approx k \log(\Omega(E)\cancel{\Delta E})$$

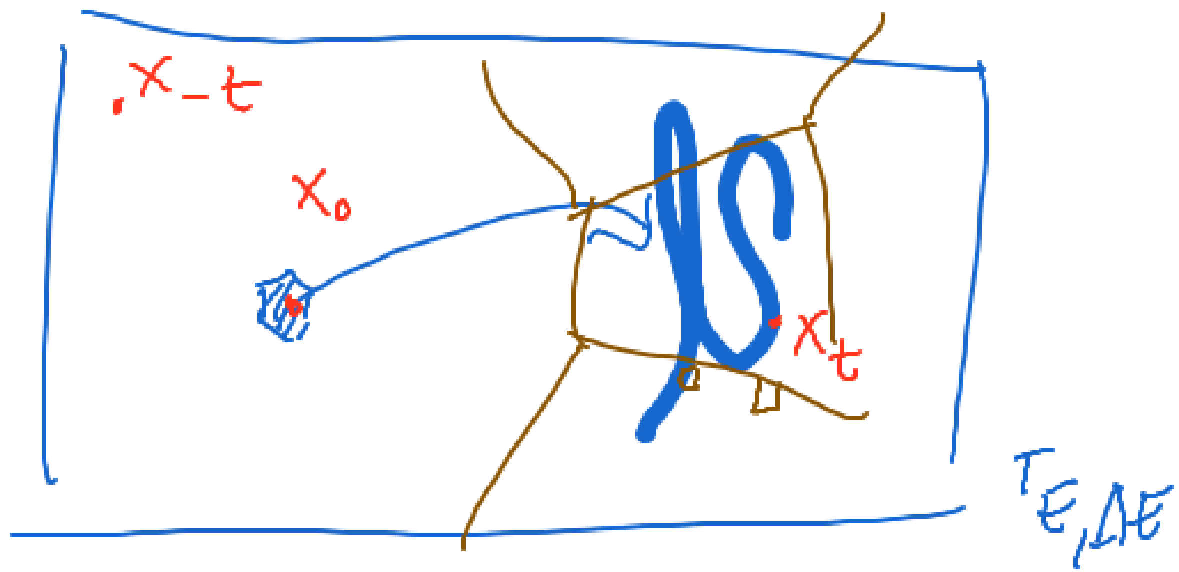
Clausius: $dS = \dfrac{dQ}{T}$

$$\Gamma_v = \Gamma_{v_A} \times \Gamma_{v_B}$$

$$\log \mathrm{vol}(\Gamma_v) = \log \mathrm{vol}_A(\Gamma_{v_A}) + \log \mathrm{vol}_B(\Gamma_{v_B})$$

2nd law

typicality argument:



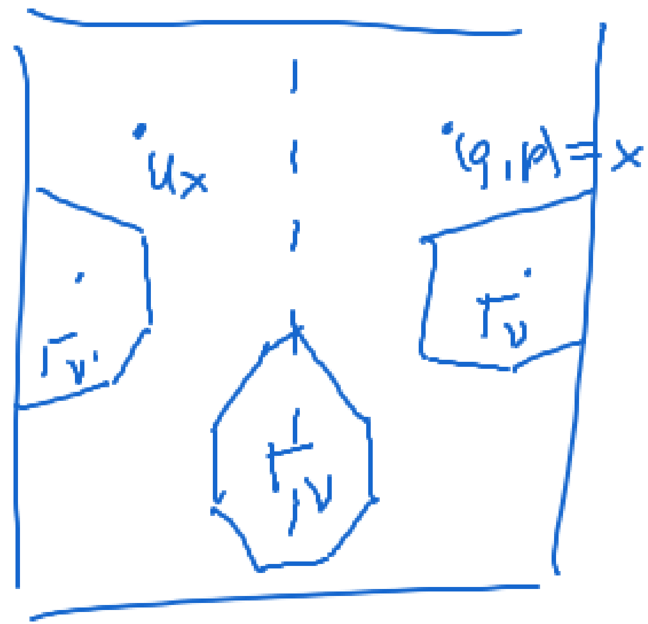
shallow = small depth
short-lived = small lifetime
infrequent

or im-probable

if ergodic

Loschmidt: time reversal

$$U : \Gamma \to \Gamma$$
$$(q,p) \mapsto (q,-p)$$



Γ

Rule of macro time reversal

$$\forall \nu \ \exists \bar\nu : \ U\Gamma_\nu = \Gamma_{\bar\nu}.$$

Γ

Examples of Macro Variables

1) Hydrodynamic variables

hydrodynamic eq.s : Euler eq.s

Navier - Stokes eq.s

(PDEs)

$n(\underline{r}, t)$ $\underline{r} \in R^3$ or Λ, $t \in R$

particle no. density

$e(\underline{r}, t)$

energy density

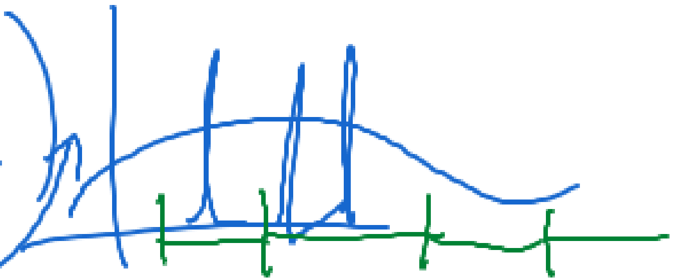
$\underline{p}(\underline{r}, t)$

momentum density

$$n_x(\underline{r}) = \sum_{j=1}^{N} \delta^3(\underline{r} - q_j)$$

emp. dist.

$$e_x(\underline{r}) = \sum_{j=1}^{N} \left(\frac{p_j^2}{2m} + \ldots \right) \delta^3(\underline{r} - q_j)$$



$$p_x(\underline{r}) = \sum_{j=1}^{N} p_j\, \delta^3(\underline{r} - q_j)$$

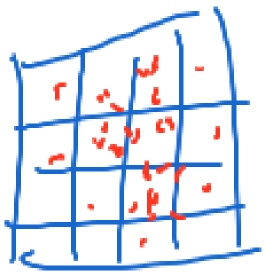
+ coarse grain.

+ rounding.

2) Boltzmann considered

- choose partition of Γ_1 ($\in \Lambda \times \mathbb{R}^3$)

$A_1 \dots A_\nu$



Γ_1

$N_i = \#\{ j = 1 \dots N : x_j \in A_i \}$

occupation no. of A_i

occ. fraction

$$f_i = \frac{N_i}{N}$$

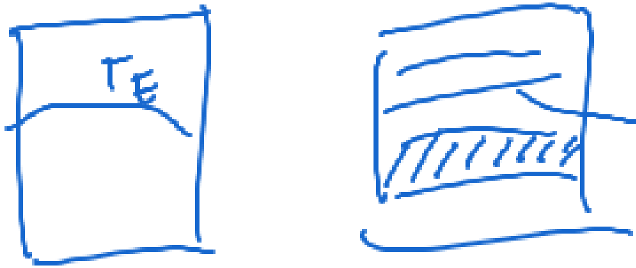
another

- coarse-graining $\hat{=}$ rounding

macroscopic resolution Δf (e.g. 10^{-10})

(rounded f_i) $= M_i$

3) toy ex: $A \cup B$



E_A = macro variable = ν

Γ Γ_E $E_B = E - E_A$

$$\lambda_E(\Gamma_E) = \Omega(E) = \int_0^E dE_A \, \Omega_A(E_A)\, \Omega_B(E - E_A)$$

$$\mathrm{vol}(\Gamma_\nu) = \Omega_A(E_A)\, \Omega_B(E - E_A)\, \Delta E \; \cancel{+ o(\Delta E)}$$

Find the largest T_V: $\quad$ max! = vol T_V

$$\Omega_A(E_A)\,\Omega_B(E-E_A) = \max!$$

$$0 = \frac{\partial}{\partial E_A}\Big(\qquad \Big)$$

$$= \Omega_A'(E_A)\,\Omega_B(E-E_A) - \Omega_A(E_A)\,\Omega_B'(E-E_A)$$

$$\Longleftrightarrow \quad (\log \Omega_A)' = \frac{\Omega_A'}{\Omega_A}(E_A) = \frac{\Omega_B'}{\Omega_B}(E-E_A) = (\log \Omega_B)'$$

$$\log \Omega_i(E) = \frac{1}{k} S_i(E) \qquad i = A, B$$

$$\Rightarrow \quad \frac{\partial S_A}{\partial E_A}(E_A) = \frac{\partial S_B}{\partial E_B} \quad (E_B = E - E_A)$$

$$\Rightarrow \quad \frac{1}{T_A} = \frac{1}{T_B}$$

more explicitly: ideal gas $\Rightarrow \dfrac{3kN_A}{2E_A} = \dfrac{3kN_B}{2(E-E_A)}$

$N = N_A + N_B$

Unique sol: $\hat{E}_A = \dfrac{N_A}{N} E$

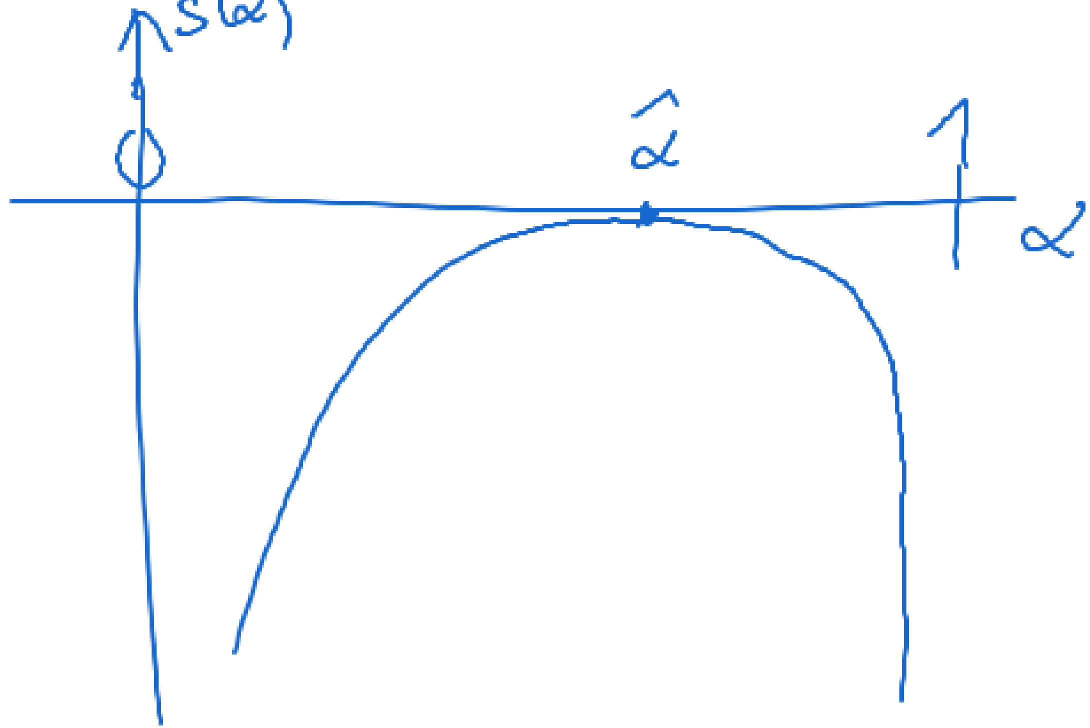
2nd derivative test $\Rightarrow$ ... $\Rightarrow$ strict global max.

$$\frac{\operatorname{vol}\Gamma_\nu}{\operatorname{vol}\Gamma_{\hat{\nu}}} = \exp\left(\frac{1}{k}\left(S(E_A) - S(\hat{E}_A)\right)\right)$$

$$= \exp\left(\frac{3N}{2} s(\alpha)\right), \quad s = \frac{S}{N} \text{ entropy per particle}$$

$\alpha = E_A/E$

$$s(\alpha) = \hat{\alpha} \log \frac{\alpha}{\hat{\alpha}} + (1-\hat{\alpha}) \log \frac{1-\alpha}{1-\hat{\alpha}}.$$



dramatic difference in size!

4) <u>Ex</u> Estimate $\frac{\mathrm{vol}\,\Gamma_{eq}}{\mathrm{vol}\,\Gamma_{mc}} \simeq 1 - \exp(-10^{-15} N)$.