Gibbs entropy

$$s(f) = -k \sum_{i=1}^{r} \Delta_i f_i \log f_i \qquad f = (f_1 \dots f_r)$$

Found: MBD $\quad f_i = \frac{1}{Z} e^{-\beta H_i}$

maximizes $s(f)$ under constraints

$$\sum_i f_i = 1$$

$$\sum_i f_i H_i = E$$

continuum limit

$$f : \Gamma_1 \to \mathbb{R}$$

$$S[f] = -k \int_{\Gamma_1} dx_1 \, f(x_1) \log f(x_1)$$

MBD: $f(x_1) = \frac{1}{Z} e^{-\beta H_1(x_1)}$

maximizes $S[f]$ under $\int f(x_1) = 1$

and $\int f(x_1) H_1(x_1) = e$

same in Γ :

$$\rho_{can}(x) = \frac{1}{Z} e^{-\beta H(x)}$$

maximizes, under $\int_\Gamma \rho(x) = 1$, $\int_\Gamma \rho(x) H(x) = E$,

$$S_{Gibbs}[\rho] = -k \int_\Gamma dx\, \rho(x) \log \rho(x) .$$

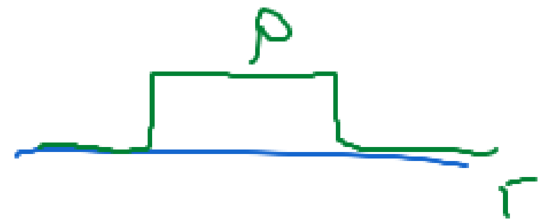
Gibbs entropy

Properties of S_{Gibbs}

1) If $\Delta \subset \Gamma$, flat $\rho(x) = \frac{1}{\mathrm{vol}\,\Delta} 1_{x\in\Delta}$.

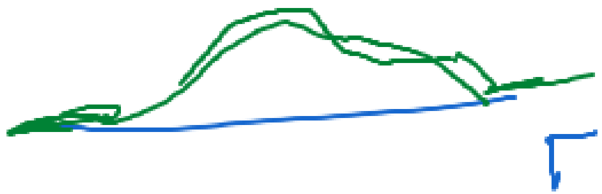
then $S_{Gibbs}[\rho] =$

$$= -k \int_{\cancel{\Gamma}\,\Delta} dx \, \frac{1}{\mathrm{vol}\,\Delta} \cancel{1_\Delta} \log\left(\frac{\cancel{1_\Delta}\; 1}{\mathrm{vol}\,\Delta}\right)$$



$$= +k \frac{1}{\cancel{\mathrm{vol}\,\Delta}} \cancel{\int_\Delta dx} \log \mathrm{vol}\,\Delta = k \log \mathrm{vol}\,\Delta$$

$= S(v)$ if $\Delta = \Gamma_v$.



2) Ex ideal gas: $\rho_{can}(x) = \frac{1}{Z} e^{-\beta H(x)}$

$$= f(x_1)\, f(x_2) \cdots f(x_N)$$

$$S_{Gibbs}[\rho_{can}] = -\frac{k}{N!} \int_{\Gamma_0} dx_1 \cdots dx_N \, f(x_1) \cdots f(x_N) \sum_j \log f(x_j)$$

$$= -\frac{k}{N!} \sum_j \int_{\Gamma_1} dx_j \, f(x_j) \log f(x_j) = \cancel{\frac{N}{N!}} \quad s[f]$$

$$\propto \quad S(eq) = k \log \operatorname{vol} \Gamma_{eq} .$$

That is, $\underline{S_{Gibbs}[\rho_{can}]} = S(eq) = S\underline{(N,E,V)}$

$$= S_{Gibbs}[\rho_{eq} \, 1_{\Gamma_{eq}}] \approx \underline{S_{Gibbs}[\rho_{mc}]}$$

equiv. of ensembles

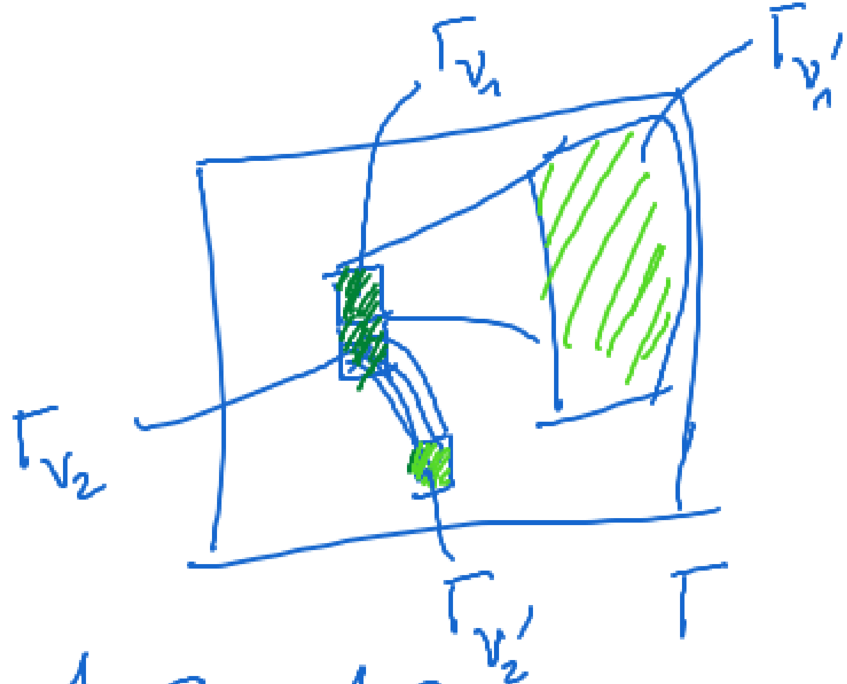
3) definition of entropy? Controversial!

subjective ρ represents knowledge of an observer.

Ex wrong values

case 1: switch on

case 2: switch off



$$S_1 > S_0 = S_2$$

$$S_{Gibbs}[\rho] = \frac{1}{2} S_{Gibbs}(\mathbb{1}_{v_1'}) + \frac{1}{2} S_{Gibbs}(\mathbb{1}_{v_2'}) \pm \cancel{4\,\frac{1}{2}\log\frac{1}{2}} = \frac{1}{2} S_1 + \frac{1}{2} S_2$$

4) time evolution: $S_{Gibbs}[\rho_t] = \text{const.}$

if ρ_t evolves according to phase space evol.

Ensemblist vs. individualist views

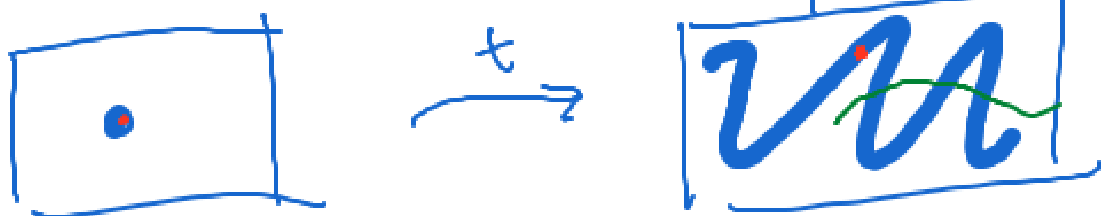
of thermal eq.

~~[illegible]~~ definition of th. eq.

1) ensemblist: $\rho = \rho_{mc}$ or $\rho = \rho_{can}$.

2) individualist: $x \in \Gamma_{eq}$

mixing:



weak convergence: $\rho\ \Gamma$ $\qquad \rho_t \xrightarrow{t\to\infty} \Gamma$ uniform

X complete separable metric space

Def μ_n

$$\text{w-}\lim \mu_n = \mu \;:\Longleftrightarrow$$

$\forall f : X \to \mathbb{R}$ bdd and cont.:

$$\int_X f(x)\, \mu_n(dx) \xrightarrow{n \to \infty} \int_X f(x)\, \mu(dx)$$

Thms about systems with interaction in the thermodynamic limit

$E, V, N \to \infty$, $\frac{E}{N} \to e$, $\frac{V}{N} \longrightarrow v$

$e, v \in (0, \infty)$, $\Lambda_N \subset \mathbb{R}^3$. Two types of thms:

1) In the th. limit, $S(E, N, V) = k \log \Omega(E)$ is asymptotically $S = N s(e, v) + o(N)$.
$s(e, v)$ is concave, indep. of the shape of Λ_N.
And equiv. of ensembles: $S_{Gibbs}[\rho_{can}] = N s(e, v) + o(N)$.
for suitable $\beta = \beta(e, v)$.

2) In the th. limit, $\exists$ dominant macro state; using N-indep. description ν, asympt. $S(\nu) = N\, s(\nu) + o(N)$.

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1) D. Ruelle (1963)
M. Fisher (1964) } short range int.

J. Lebowitz &
E. Lieb (1969) } Coulomb int

2) O. Lanford (1973)