Mean Free Path / Time

$d^3\underline{q}\, d^3\underline{v}$, no. of particles hitting it in dt

with velocity in $d^3\underline{v}_*$ and collision parc in $d^2\underline{\omega}$

is $N f(\underline{q}, \underline{v}_*)\, 1_{\underline{\omega}\cdot(\underline{v}-\underline{v}_*)>0}\, \underline{\omega}\cdot(\underline{v}-\underline{v}_*)\, dC\, d^3\underline{v}_*$

with $dC = 4a^2 d^2\underline{\omega}$ $\quad$ dt

Thus, prob. of collision during dt is

$$4Na^2 dt \int_{\mathbb{R}^3} d^3\underline{v}_*\, f(\underline{q}, \underline{v}_*) \int_{S^2} d^2\underline{\omega}\; 1_{\underline{\omega}\cdot(\underline{v}-\underline{v}_*)>0}\, \underline{\omega}\cdot(\underline{v}-\underline{v}_*)$$

$$= 4\pi \cancel{f} N a^2 dt \int_{\mathbb{R}^3} d^3\underline{v}_* \, f(q, \underline{v}_*) \, |\underline{v} - \underline{v}_*| \, \cancel{dt}.$$

If $\cancel{f}$ $\frac{\partial f}{\partial t}$ small,

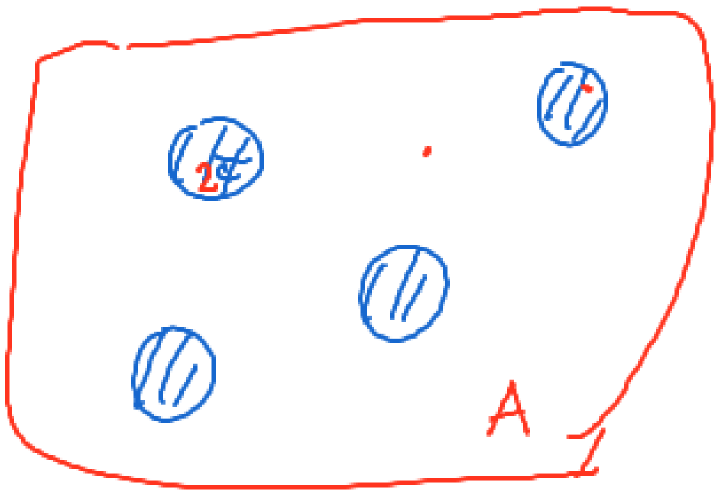
$$\tau = \text{mean free time} = \tau(q, \underline{v})$$

$$= \left[4\pi N a^2 \int d^3\underline{v}_* \, f(q, \underline{v}_*) \, |\underline{v} - \underline{v}_*| \right]^{-1}$$

average further over all particles

For average mean free path ℓ,

Boltzm.'s heuristic argument:



$$\text{Prob(hitting)} = \frac{4\pi a^2 N_L}{A} \qquad N_L = \frac{NAL}{V}$$

$\Rightarrow$ $\mathbb{E}$ no. of collision while travelling distance L

$$= \cancel{\frac{A \quad V}{4\pi a^2 NAL}} \quad \frac{4\pi a^2 N\cancel{A}L}{\cancel{A}V}$$

so free path $= \dfrac{\cancel{4\pi}\, V}{4\pi a^2 N} \xrightarrow[\text{BG limit}]{N\to\infty} \dfrac{V}{\pi\lambda} = \ell$

$$\Rightarrow \bar{t} = \frac{\ell}{v_{rms}} = \left(\frac{m}{3kT}\right)^{1/2} \frac{V}{\pi\lambda}$$

mean free time

Properties of the Boltzm. Eq.

1) Conservation laws:

a) $$\int_{\Lambda} d^3q \int_{\mathbb{R}^3} d^3\underline{v}\, f(q,\underline{v},t) = 1 \quad \forall t$$

b) energy per particle
$$\int_{\Lambda} d^3q \int_{\mathbb{R}^3} d^3\underline{v}\, \underline{v}^2 f(q,\underline{v},t)$$ is conserved.

2) The H theorem

__Def__ H functional: $H(f) = \int_\Lambda d^3q \int_{\mathbb{R}^3} d^3\underline{v} \; \overbrace{f(q,\underline{v}) \log f(q,\underline{v})}$

$s(f) = -k\,H(f)$

__Claim__ If f_t is a sol. of the Boltzm. eq. then

$$\frac{dH(f_t)}{dt} \leq 0.$$

Derivation: $f' = f(q, \underline{v}')$, $f_* = f(q, \underline{v}_*)$, f_*'

Prop 4 For $f(q, \underline{v})$ and $\phi(q, \underline{v})$

$$\int_{\mathbb{R}^3} d^3\underline{v}\, Q\, \phi = \frac{\lambda}{4} \int d^3\underline{v} \int d^3\underline{v}_* \int d^2\underline{\omega}\; 1_{\underline{\omega}\cdot(\underline{v}-\underline{v}_*)>0} \times$$

$$\times\, \underline{\omega}\cdot(\underline{v}-\underline{v}_*)\left(f' f_*' - f f_*\right)\left(\phi + \phi_* - \phi' - \phi_*'\right)$$

Pf: $\int d^3\underline{v}\, Q\phi = \lambda \int d^3\underline{v} \int d^3\underline{v}_* \int d^2\underline{\omega}\; 1\; \underline{\omega}\cdot(\underline{v}-\underline{v}_*)\left(f' f_*' - f f_*\right)\phi_*$

rename $\underline{v} \longleftrightarrow \underline{v}_*$, $\underline{\omega} \to -\underline{\omega}$, likewise with ϕ' $\square$

<u>Prop 5</u> "Boltzmann inequality": If $f(q,\underline{v}) > 0$

then $\int d^3\underline{v}\, Q \log f \le 0$

<u>Pf</u>: 1) Use Prop 4 with $\phi = \log f$, so

$$\int d^3\underline{v}\, Q \log f = \frac{\lambda}{4} \int d\underline{v}\, d\underline{v}_* d\underline{\omega}\; \underline{1}\; \underline{\omega}\cdot(\underline{v}-\underline{v}_*) \underbrace{\left(\underbrace{f'f'_*}_{z} - \underbrace{ff_*}_{y}\right) \log \frac{\overset{y}{ff_*}}{\underset{z}{f'f'_*}}}_{\le 0}$$

2) For $y, z > 0$ $\quad (z-y) \log \frac{y}{z} \le 0$

Because if $y<z$ OK, if $y=z$ OK, if $y>z$ OK.

So, integrand ≤ 0, so integral ≤ 0. $\square$

Last step: derivation of the H thm

$$\frac{dH}{dt} = \frac{d}{dt}\int d^3\underline{q}\int d^3\underline{v}\; f \log f = \int d^3\underline{q}\, d^3\underline{v}\; \frac{\partial}{\partial t}\left(f \log f\right)$$

$$\partial\left(f \log f\right) = \left(\partial f\right) \log f + \cancel{f}\,\frac{\partial f}{\cancel{f}}$$

$$= \left(\partial f\right)\left(1 + \log f\right)$$

B. Eq: $\left(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla_{\underline{q}}\right) f = Q$

$$\Rightarrow \left(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla_{\underline{q}}\right)\left(f \log f\right) = \left[\left(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla_{\underline{q}}\right) f\right]\left(1 + \log f\right)$$

$$= Q(1+\log f)$$

$$\int d\underline{q}\, d\underline{v}\, \frac{\partial f}{\partial t}(f \log f) = -\int d\underline{q}\, d\underline{v}\, (\underline{v}\cdot\nabla_{\underline{q}}(f))(1+\log f) + \underbrace{\int d\underline{v}\, Q(1+\log f)}_{\underbrace{\int d\underline{v}\, Q}_{0 \text{ by Prop 4}} + \underbrace{\int d\underline{v}\, Q \log f}_{\leq 0 \text{ by Prop 5}}}$$

$$\frac{dH}{dt} = -\underbrace{\int d\underline{q}\, d\underline{v}\, (\underline{v}\cdot\nabla_{\underline{q}} f)(1+\log f)}_{\text{will show } = 0} + \text{sth} \leq 0$$

$$\text{term} = \int d\underline{q} \int d\underline{v} \underbrace{(\underline{v} \cdot \nabla_{\underline{q}} f)(1 + \log f)}_{\underline{v} \cdot \nabla_{\underline{q}} (f \log f)}$$

$$= \int_{\Lambda} d^3\underline{q}\ \nabla_{\underline{q}} \cdot \int d\underline{v}\ \underline{v}\, f \log f$$

$$\boxed{f(\underline{q}, \underline{v} - 2(\underline{n} \cdot \underline{v})\underline{n}) = f(\underline{q}, \underline{v})}$$

Gauss thm

$$= \int_{\partial\Lambda} d^2\underline{q}\ \underline{n}(\underline{q}) \cdot \overbrace{\int_{\mathbb{R}^3} d^3\underline{v}\ \underline{v} \cdot \underline{n} f \log f}^{0}$$

$$= \cancel{\int d\underline{v} \overbrace{\int_{\partial\Lambda} d^2\underline{q}\ \underline{n} \cdot \underline{v}\ f \log f}^{0}} = 0. \quad \square$$

3) Maxwellians

Def local Maxwellians

$$f(q,\underline{v}) = n(q) \left(\frac{\beta(q)m}{2\pi}\right)^{3/2} \exp\left[-\beta(q)\frac{m}{2}(\underline{v}-\underline{u}(q))^2\right]$$

$\uparrow$
no. density

global Maxwellians : same with q-indep.

$n, \beta, \underline{u}$, $\underline{u}=0$.

"corresponding" global Maxw for f, $M(q,\underline{v}) = c_1 e^{-c_2 v^2}$

so that $\int d^3q\, d^3v\, M = 1$ and $\int d^3q\, d^3v\, \underline{v}^2 M = \int d^3q\, d^3\underline{v}\; \underline{v}^2 f$

Problem

Conj as $t \to \infty$, $f_t \longrightarrow M$ corresp. glob. Maxw.

Open problem, partial results.

Prop. 6 If $f \in C^2$, $f > 0$, then $\frac{dH}{dt} = 0 \Longleftrightarrow$ f is a local Maxw. That is, all stationary sol.s are ~~loc.~~ glob. Maxw.s.

A loc. Maxw. on $\Lambda \times \mathbb{R}^3$ satisfying the B.C. is stationary iff it is a global Maxw. $\Longrightarrow$

Prop 7 Let M be the corr. glob. Maxw. for f. Then $H(f) \geq H(M)$.

Prop 8 If $H(f_t) \longrightarrow H(M)$ as $t \to \infty$

then $f_t \longrightarrow M$ in L^1

i.e. $\int d^3q\, d^3v \, |f_t - M| \longrightarrow 0$.

Thm 17 (Desvillettes and Villani 2004)

Suppose f_t sol. of the Boltzm. Eq $\forall t \geq 0$,

f_t is very regular uniformly in t

f_t is bdd away from 0

$$f_t(q,\underline{v}) \geq K e^{-A|\underline{v}|^n}.$$

Then $f_t \to M$ in L^2 and in all Sobolev norms.

Existence of Solutions

Thm 18 Λ is compact and has smooth $\partial\Lambda$, initial datum f_0 is cont., satisfies the b.c., normalized, satisfies the bound

$$f_0(q,\underline{v}) \le z \left(\frac{m\beta}{2\pi}\right)^{3/2} e^{-\beta m \underline{v}^2/2}$$

for some $z,\beta > 0$. Then a "mild sol." f_t of the B. Eq. exists for $t \in [0, \frac{1}{5}\bar{t}]$. The sol. is unique, $f_t \ge 0$.