Recurrence

occurs in QM for both pure and mixed states, finite- or infinite-dim $\mathcal{H}$

<u>Prop 10</u> Suppose H has pure point spectrum. For any $\rho_0 \geq 0$ with $\mathrm{tr}\,\rho_0 = 1$, any $\varepsilon > 0$, and any $T > 0$, there is $t > T$ such that

$$\| \rho_t - \rho_0 \|_{tr} < \varepsilon .$$

with $\rho_t = e^{-iHt} \rho_0 e^{iHt}$.

Ex no recurrence:

$H = -\Delta$, $\mathcal{H} = L^2(\mathbb{R}^d, \mathbb{C})$

spectrum $= [0, \infty)$

wave fct propagates to spatial ∞.

Thermodynamic Ensembles

Micro-canonical ensemble

$$I_{mc} = (E - \Delta E, E], \quad H = \sum_\alpha E_\alpha |\phi_\alpha\rangle\langle\phi_\alpha|$$

subspace $\mathcal{H}_{mc} = \overline{\text{span}} \{ \phi_\alpha : E_\alpha \in I_{mc} \}$

$= \text{spectral subspace of } I_{mc}$

$= \operatorname{ran} \mathbb{1}_{I_{mc}}(H)$

micro-canonical subspace, energy shell

"most ψ": uniform measure over ψ

u_{mc} on $\mathbb{S}(\mathcal{H}_{mc})$, $\underbrace{\dim \mathcal{H}_{mc}}_{d_{mc}} < \infty$.

$$\rho_{mc} := \rho_{u_{mc}} = \frac{1}{d_{mc}} P_{mc} = \frac{1}{d_{mc}} 1_{I_{mc}}(H)$$

"micro-can density matrix" = "micro-can. ensemble"

Density of States

$[E-dE, E]$

$$\Omega(E)\, dE \approx \#\{\alpha : E_\alpha \in [E, E+dE)\}$$

$\Omega : \mathbb{R} \to \mathbb{R}$ cont., approximative, on a certain scale

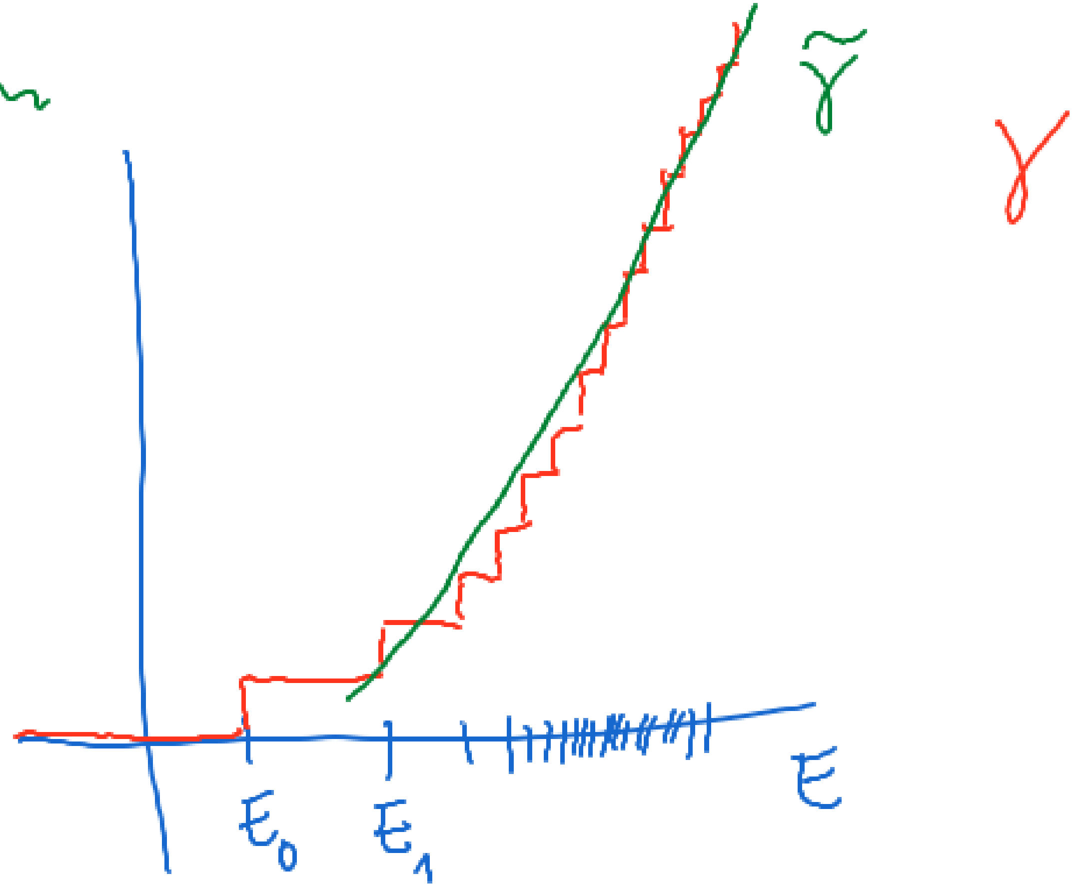
$$\gamma(E) := \text{dim spectral subspace of } (-\infty, E]$$

$$= \operatorname{tr} 1_{(-\infty, E]}(H)$$

$$= \sum_{\alpha : E_\alpha \le E} m_\alpha$$

multiplicity

$\tilde{\gamma} :=$ smooth version of γ



$$\Omega(E) = \frac{d\tilde{\gamma}(E)}{dE}$$

density of states

thermal equilibrium entropy

$$S(E,V,N) = k \log \Omega(E)$$

<u>Ex</u> ideal Bose gas: N identical bosons in

$$\mathbb{R}^3 \supset \Lambda = [0,L]^3, \quad H = -\frac{\hbar^2}{2m}\Delta \quad \text{in } \mathbb{R}^{3N}$$

Dirichlet boundary condition

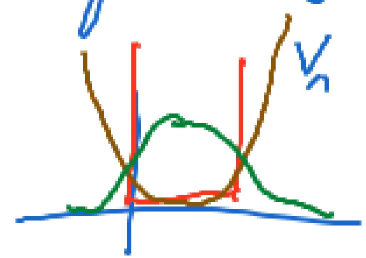
$$\psi(q) = 0 \quad \text{whenever } q_{ia} = 0 \text{ or } q_{ia} = L.$$

for ~~some~~ any $i \in \{1 \dots N\}$

and any $a \in \{1 \dots 3\}$

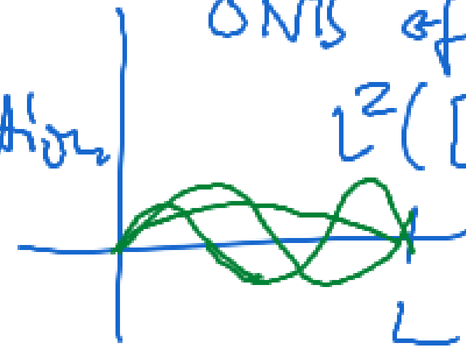
$$q = (q_1 \dots q_N)$$

$$V_1(q) = \begin{cases} 0 & \text{if } q \in \Lambda \\ \infty & \text{if } q \notin \Lambda \end{cases}$$



1d: $E_n = \frac{\pi^2}{2mL^2} n^2, \quad \varphi_n(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(n \frac{\pi}{L} q\right)$

ONB of $L^2([0,L])$



3Nd: first leave aside symmetrization

$$\mathcal{H} = L^2([0,L]^{3N})$$

$$\varphi_n(q) = \left(\frac{2}{L}\right)^{\frac{3N}{2}} \prod_{i=1}^{N} \prod_{a=1}^{3} \sin\left(n_{ia} \frac{\pi}{L} q_{ia}\right)$$

$$n = (n_1 \dots n_{N3}) = (n_{ia})_{ia}$$

$$E_n = \sum_{i=1}^{N} \sum_{a=1}^{3} \frac{\pi^2}{2mL^2} n_{ia}^2 = \frac{\pi^2}{2mL^2} \|n\|^2$$

$$7^2 + 4^2 = 8^2 + 1^2$$

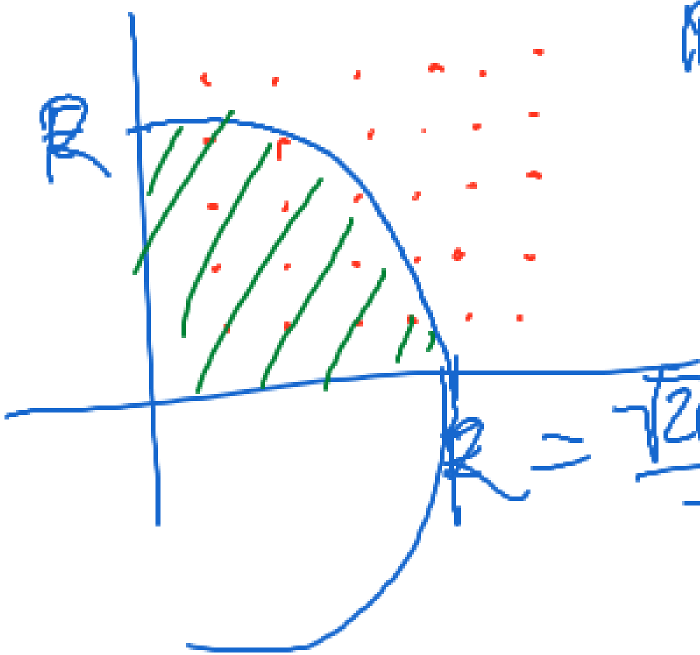
Now: $\mathcal{H}_+$ = permutation - symmetric subspace

Claim $\{ c_n P_+ \varphi_n : n \in \mathbb{N}^{3N} \}$ is an ONB of $\mathcal{H}_+$ consisting of eigenvectors of H.

Now: count

$E_n = \text{const.}\ \|n\|^2 \le E \qquad n = (n_{11} \ldots n_{N3}) \in \mathbb{N}^{3N}$

$\mathbb{R}^{3N}$



$R = \frac{\sqrt{2m}\,L}{\pi}\sqrt{E}$

$\mathrm{Vol}(\text{ball}) = \frac{\pi^{3N/2}}{(3N/2)!} R^{3N}$

$\#\text{quadrants} = 2^{3N}$

$$\#\{n : E_n \le E\} \approx \frac{\mathrm{Vol}(\text{ball})}{2^{3N}} = \frac{1}{(3N/2)!}\left(\frac{\pi R^2}{2^2}\right)^{3N/2}$$

$$= \frac{1}{(3N/2)!}\left(\frac{\cancel{2}mEL^2}{2\cancel{4}\pi}\right)^{3N/2}$$

[using $k! \approx k^k e^{-k}$]

$$\approx \frac{1}{\left(\frac{3N}{2}\right)^{3N/2} e^{-3N/2}} \left(\frac{mEL^2}{2\pi} \right)^{3N/2}$$

$$\approx \left(\frac{mEL^2 e}{3\pi N} \right)^{3N/2}$$

$$\gamma(E) \approx \frac{1}{N!} \left(\quad " \quad \right)^{3N/2} \approx \left(\frac{mEL^2 e^{5/3}}{3\pi N^{5/3}} \right)^{3N/2} =: \tilde{\gamma}(E)$$

so $\Omega(E) = \frac{3N}{2}\left(\quad \cdots \quad \right)^{3N/2 - 1}$

leading order (neglect factors of order N^{const})
in N

$V = L^3$, so $\Omega(E) \approx \left(\frac{m e^{5/3}}{3\pi} \frac{E}{N}\right)^{3N/2} \left(\frac{V}{N}\right)^N$

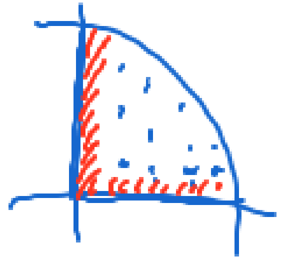
so $S(E, N, V) = kN\left[\frac{3}{2}\log\frac{E}{N} + \log\frac{V}{N} + \frac{3}{2}\log\frac{4\pi m}{3} + \frac{5}{2} - 3\log(2\pi\hbar)\right]$

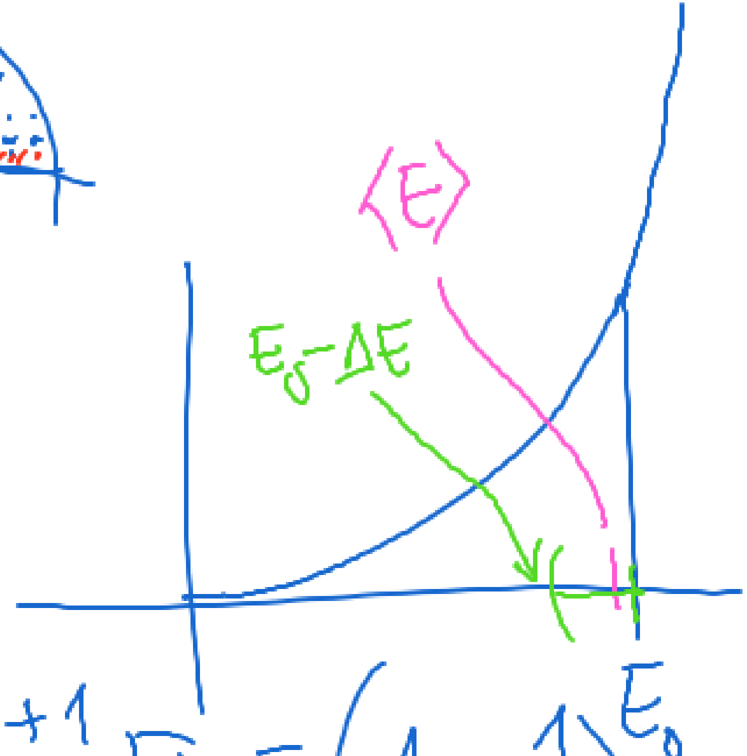
$= S_{cl} - kN3\log(2\pi\hbar)$.

Corrections:

1) permutation
2) sphere
3) plane

$n_{ia} \in \mathbb{N}$





Ex $\qquad \Omega(E) \propto E^{3N/2 - 1}$

$$p(E) = 1_{0<E<E_0} (\alpha+1) E_0^{-\alpha-1} E^{\alpha}$$

has $\langle E \rangle = \int_0^{E_0} dE \, E \, p(E) = \frac{\alpha+1}{\alpha+2} E_0 = \left(1 - \frac{1}{\alpha+2}\right) E_0$