The canonical density matrix

Def $\rho_{can} = \frac{1}{Z} e^{-\beta H}$ $\qquad \beta = \frac{1}{kT}$

For $\operatorname{tr} \rho_{can} = 1$, need $\operatorname{tr} e^{-\beta H} < \infty$

$\iff \sum_\alpha \exp(-\beta E_\alpha) < \infty$

$H = \sum_\alpha E_\alpha |\alpha\rangle\langle\alpha|$

__Rule__ Suppose that $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_b$, that b is much larger than s, that the interaction between s and b is negligible, $H = H_s \otimes I_b + I_s \otimes H_b$

Then $\operatorname{tr}_b \rho_{\substack{mc \\ sub}} \approx \rho_{can}$.

__Derivation__ for b = ideal gas, actually

$$\Omega_b(E) = (\text{const}) \left(\frac{E}{N_b}\right)^{3N_b/2}$$

Let $I_{mc} = (E - \Delta E, E]$, suppose H_s, H_b are non-degenerate,

$V_b = N\bar{v}.$ $\qquad H_a = \sum_\alpha E_\alpha |\alpha\rangle\langle\alpha| \qquad \alpha = (a, n)$

$a = s \qquad H_b = \sum_\beta E_\beta |\beta\rangle\langle\beta| \qquad \beta = (b, n)$

$\rho_a = \mathrm{tr}_b\, \rho_{mc}$

$\rho_{mc} = \mathcal{N}_{N_b} \sum_{\alpha,\beta} 1_{E_\alpha + E_\beta \in I_{mc}} \qquad |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|$

$$\rho_a = \mathcal{N}_{N_b} \sum_\alpha \# \{ \beta : \underbrace{E_\alpha + E_\beta \in I_{mc}}_{\underbrace{E_\beta \in I_{mc} - E_\alpha}_{\approx \Omega_b(E - E_\alpha)\Delta E \text{ (small } \Delta E) \text{ or } \approx \gamma_b(E-E_\alpha) \text{ (large } \Delta E)}} \} |\alpha\rangle\langle\alpha|$$

$$\approx \tilde{\mathcal{N}}_{N_b} \sum_\alpha \left(\frac{E - E_\alpha}{N}\right)^{3N/2} |\alpha\rangle\langle\alpha| \qquad [N = N_b]$$

$$= \tilde{\mathcal{N}}_N \, \bar{e}^{3N/2} \sum_\alpha \left(1 - \frac{\frac{3}{2}E_\alpha}{\frac{3}{2}N\bar{e}}\right)^{3N/2} |\alpha\rangle\langle\alpha| \qquad [\bar{e} = E/N]$$

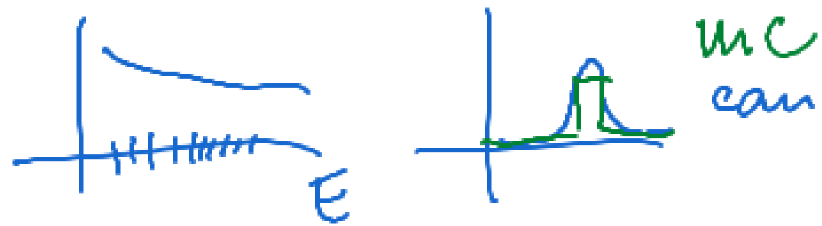
$$\xrightarrow{N\to\infty} \frac{1}{Z}\sum_\alpha e^{-\frac{3}{2}\frac{E_\alpha}{\bar e}} |\alpha\rangle\langle\alpha|$$

i.e., $\rho_a \overset{N\to\infty}{=} \frac{1}{Z} e^{-\beta H_a}$ with $\beta = \frac{3}{2\bar e}$

$$\Leftrightarrow \bar e = \frac{3}{2}kT. \qquad \square$$

<u>Rem</u> equiv. of ensembles "$\rho_{can} \approx \rho_{mc}$".

both diag. in $\{|\alpha\rangle\}$,



Heisenberg Model = quantum Ising model

$\Lambda \subset \mathbb{R}^d$, $\# \Lambda = N$, $\mathcal{H} = (\mathbb{C}^2)^{\otimes N} \ni \psi_{s_1 \dots s_N}$

$$H = -J \sum_{\underline{i},\underline{j} : |\underline{i}-\underline{j}| = 1} \underline{\sigma}_{\underline{i}} \cdot \underline{\sigma}_{\underline{j}} , \qquad \underline{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$$

Pauli matrices

consider a with $\rho_{can} = \frac{1}{Z} e^{-\beta H}$,

difference to Ising model: time evolution e^{-iHt}

no symmetrization needed.

Canonical typicality:

For most pure states ψ ~~$\not\in$~~ from an energy shell $\mathcal{H}_{mc}$ of a large system, the reduced density matrix of a small subsystem s is approx. canonical,

$$\underbrace{\rho_s^{\psi}}_{:= \mathrm{tr}_{s^c} |\psi\rangle\langle\psi|} \approx \rho_{can}$$

$b := s^c$

Previous $\mathrm{tr}_b\, \rho_{mc} \approx \rho_{can}$. Now $\mathrm{tr}_b |\psi\rangle\langle\psi| \approx \rho_{can}$

for most $\psi \in \mathbb{S}(\mathcal{H}_{mc})$

$\psi \mapsto \rho_s^\psi$ is nearly const.

$\psi \sim u_{mc}$ uniform on $\mathbb{S}(\mathcal{H}_{mc})$

then $\underline{\underline{\mathbb{E}\, \rho_s^\psi}} = \int_{\mathbb{S}(\mathcal{H}_{mc})} u_{mc}(d\psi)\, \rho_s^\psi$

$$= \int_{\mathbb{S}(\mathcal{H}_{mc})} u_{mc}(d\psi)\, \mathrm{tr}_b\, |\psi\rangle\langle\psi| = \mathrm{tr}_b \int_{\mathbb{S}(\mathcal{H}_{mc})} u_{mc}(d\psi)\, |\psi\rangle\langle\psi|$$

$$= \operatorname{tr}_b \; \rho_{mc} \, .$$

<u>Comparison to classical:</u>

sub, $X_{sub} = (X_s, X_b)$

"pure state"

Quantum: sub: ψ, ρ_s^ψ mixed unless $\psi = \psi_s \otimes \psi_b$

size: think of Heisenberg model: $d_s = \dim \mathcal{H}_s$
$= 2^{N_s}$

$(\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_b \supset \mathcal{H}_{mc})$

<u>Thm 20</u> Let $\mathcal{H}_s$ and $\mathcal{H}_b$ be Hilbert spaces of dim d_s and d_b. Set $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_b$, $\mathcal{H}_{mc}$ be any subspace of $\mathcal{H}$ of dim d_{mc}, $\rho_{mc} = \frac{1}{d_{mc}} P_{mc}$, u_{mc} uniform on $\mathbb{S}(\mathcal{H}_{mc})$. Then

$$\forall \varepsilon > 0 \quad u_{mc}\left\{ \psi \in \mathbb{S}(\mathcal{H}_{mc}) : \left\| \rho_s^\psi - \mathrm{tr}_b\, \rho_{mc} \right\|_{tr} < \varepsilon \right\} \geq 1 - \frac{d_s^4}{\varepsilon^2 d_{mc}}.$$

"most" $(n_{mc} \simeq 1)$ $\iff$ $\dfrac{d_s^4}{\varepsilon^2 d_{mc}} \ll 1$

$d_s^4 \ll \varepsilon^2 d_{mc}$ for reasonably small ε

$\iff$ $d_s \ll d_{mc}^{1/4}$

<u>Rem</u> 1) better estimate due to Popescu, Short, Winter (2005) $\Rightarrow$ requires only $d_s \ll d_{mc}^{1/2}$

2) What if $d_s = \infty$?

dim $=\infty$ $\qquad$ dim $<\infty$

$$\mathcal{H}_s \otimes \mathcal{H}_b \supset \mathcal{H}_{mc}$$

$$\frac{1}{Z} \tilde{P}_s \, tr_b \, \rho_{mc} \, \tilde{P}_s \qquad \sum_\alpha e^{-\beta E_\alpha}$$

$$\tilde{\mathcal{H}}_s := \text{span}\{ |\alpha\rangle : \alpha = 1 \dots n \}$$

so that $\sum_{\alpha=n+1}^{\infty} e^{-\beta E_\alpha}$ negligible

Macro states

classical: $\Gamma_{mc} = \bigcup_{\nu} \Gamma_\nu$

QM: $\mathcal{H}_{mc} = \bigoplus_{\nu} \mathcal{H}_\nu$ macro spaces

$\bigoplus$ = orthogonal sum

Def quantum Boltzmann entropy

$$S(\nu) = k \log \dim \mathcal{H}_\nu .$$

thermal eq: $\nu = eq$, $\mathcal{H}_{eq} \subset \mathcal{H}_{mc}$, $\dfrac{\dim \mathcal{H}_{eq}}{\dim \mathcal{H}_{mc}} \approx 1$

Conseq., $S(eq) = k \log \dim \mathcal{H}_{eq}$

$$\approx k \log \dim \mathcal{H}_{mc}$$

$$\approx k \log \gamma(E)$$

$$\approx k \log \Omega(E) \quad \text{(as earlier)}$$

$\mathcal{H}_\nu$ von Neumann 1929

from macro observables $M_1 \ldots M_K$.

$[M_i, M_j] = 0$, diag. simultaneously

with $\mathcal{H}_\nu$ = joint eigenspaces.