$\mathcal{H} \supset \mathcal{H}_{mc}, \quad \mathcal{H} = \bigoplus_{\nu} \mathcal{H}_{\nu}$

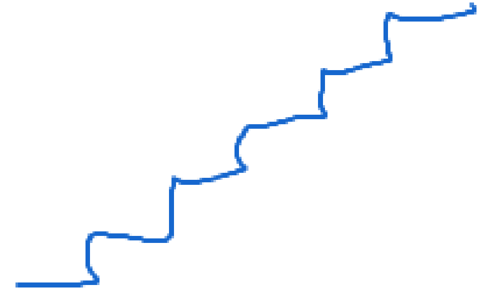
von Neumann 1929: macro observables

$M_1, \dots, M_K$ s.a. on $\mathcal{H}$

1. $[M_i, M_j] = 0$
2. Highly degenerate (order 10^N)
3. distance between eigenvalues of M_i

$=$ macro resolution of meas. of M_i.

e.g. $M_i = g(P)$



ΔM_i

spectrum $(M_i) \subset \Delta M_i \mathbb{Z}$

Then, simultaneous diag.

$$\mathcal{H}_v = \bigcap_{i=1}^{K} \mathcal{H}_{M_i, m_i}$$

$$v = (m_1, \ldots, m_K) \Rightarrow \mathcal{H} = \bigoplus_v \mathcal{H}_v$$

<u>Ex</u> (von Neum.) pos. & mom. of a macro object.

$$Q = \frac{1}{N}\sum_{i=1}^{N} Q_i$$ center-of-mass pos. op.

$$P = \sum_{i=1}^{N} P_i$$ total mom. op.

$$Q \simeq \tilde{Q}, \quad \tilde{P} \simeq P, \quad [\tilde{Q}, \tilde{P}] = 0$$

existence plausible whenever $[Q, P]$ small:

$$[Q, P] = \left[\frac{1}{N}\sum_i Q_i, \sum_j P_j\right] = \frac{1}{N}\sum_{ij}[Q_i, P_j] = \frac{1}{N}\sum_{ij}\delta_{ij}\, i\hbar$$

$= i\hbar$.

$$\Rightarrow \quad \left\| [Q, P] \right\| = \hbar \ll \underbrace{q}_{O(1)} \underbrace{p}_{O(N)} = O(N)$$

<u>Ex</u> Heisenberg model of N spins, $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$

total spin $S^z = \sum_{i=1}^{N} \sigma_i^z$, likewise S^x, S^y

~~Pr~~ $\mathcal{H} \ni \psi = \psi_{s_1 \dots s_N}$, $\sigma_i^z \psi = s_i \psi$

$$\sigma_i^z = I \otimes \dots \otimes I \otimes \underset{i\text{-th}}{\sigma^z} \otimes I \otimes \dots \otimes I$$

$$[\sigma^x, \sigma^y] = 2i\sigma^z$$

$$\Rightarrow [S^x, S^y] = \left[\sum_i \sigma_i^x, \sum_j \sigma_j^y\right] = \sum_{ij} [\sigma_i^x, \sigma_j^y]$$

$$= \sum_i 2i\sigma_i^z = 2i\, S^z$$

$$\Rightarrow \quad \|[S^x, S^y]\| = 2N \ll N^2 = \underbrace{S^x S^y}_{\|S^x\|\,\|S^y\|} =$$

$\Rightarrow$ commutator small

Natural candidate for $M_1 \ldots M_K$:

- Partition $\Lambda \subset \mathbb{R}^3$ into cells Δ_i
- For each cell, energy in Δ_i
  momentum in Δ_i
  particle no. in Δ_i
  charge in Δ_i
  total spin^z in Δ_i
- ~~coarse grain~~ find nearby commuting ops
- coarse grain (rounding)

$\longrightarrow M_1 \ldots M_K$

Math question: Is it possible to find commuting ops nearby?

I.e., $A_1 \dots A_K$ almost commute

i.e. $[A_i, A_j]$ small

$\overset{?}{\Rightarrow} \exists\, \tilde{A}_1 \dots \tilde{A}_K$, $\tilde{A}_i$ near A_i

s.t. $[\tilde{A}_i, \tilde{A}_j] = 0$

"$A_1 \dots A_K$" "nearly commuting"

$K=2$

Prop (Huaxin Lin 1995)

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall d \in \mathbb{N} \; \forall$ s.a. $d \times d$ matrices A, B:

If $\|A\| \leq 1$, $\|B\| \leq 1$, and $\|[A,B]\| < \delta$,

then $\exists \tilde{A}, \tilde{B}$ s.a. $d \times d$ matrices:

$$\|A - \tilde{A}\| < \varepsilon, \; \|B - \tilde{B}\| < \varepsilon, \; [\tilde{A}, \tilde{B}] = 0.$$

$K=3$

<u>Prop</u> (Choi 1988, Hastings & Loring 2010)

$\forall d \in \mathbb{N} \ \exists A_1, A_2, A_3$ s.a. $d\times d$ matrices:

$\|A_j\|=1$, $\|[A_i, A_j]\| \leq \frac{3}{d}$ and $\forall \tilde{A}_1, \tilde{A}_2, \tilde{A}_3$:

if commute pairwise then

$$\|A_1-\tilde{A}_1\| + \|A_2-\tilde{A}_2\| + \|A_3-\tilde{A}_3\| \geq \sqrt{1-\frac{8}{d}}$$

Explicitly, analogs of $\sigma^x, \sigma^y, \sigma^z$ Pauli

for higher spin $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$

$d = 2s+1$: $\sigma_3 = 2\begin{pmatrix} s & & & & & \\ & s-1 & & & & \\ & & s-2 & & & \\ & & & \ddots & & \\ & & & & -s+1 & \\ & & & & & -s \end{pmatrix}$

$$\sigma_1 = \begin{pmatrix} 0 & \ddots & & 0 \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & 0 \end{pmatrix},$$

$$A_i = \frac{1}{2s}\sigma_i$$

$$[\sigma_1, \sigma_2] = 2i\sigma_3$$

$$\Rightarrow [A_1, A_2] = \frac{i A_3}{s}$$

<u>Prop</u> (Ogata 2013) Let $L_1 \dots L_K$ be s.a. op.s on $\mathbb{C}^n$. For $N \in \mathbb{N}$ let $\mathcal{H}_N = (\mathbb{C}^n)^{\otimes N}$,
$L_{jk} : \mathcal{H}_N \to \mathcal{H}_N$ be L_j acting on particle k,

$$A_{jN} = \frac{1}{N} \sum_{k=1}^{N} L_{jk} .$$

Then there exist M_{jN} on $\mathcal{H}_N$ such that

$$[M_{iN}, M_{jN}] = 0 \quad \text{and} \quad \lim_{N \to \infty} \| M_{jN} - A_{jN} \| = 0 . \quad \forall j$$

M_1 = macro energy $= g(H)$

$\mathcal{H}_{mc} = \bigoplus_{\nu} \mathcal{H}_\nu$

$\nu \leftarrow$ subcollection

<u>Entropy</u> $S(\nu) = k \log d_\nu$ quantum Boltzm. entropy

<u>extensive:</u> $\mathcal{H} \otimes \mathcal{H}'$

$\mathcal{H} = \bigoplus_{\nu} \mathcal{H}_\nu$, $\mathcal{H}' = \bigoplus_{\nu'} \mathcal{H}'_{\nu'}$, interaction negligible

$$\underbrace{\bigoplus_{v,v'} \mathcal{H}_v \otimes \mathcal{H}'_{v'}}_{\mathcal{H}_{(v,v')}} = \left(\bigoplus_v \mathcal{H}_v\right) \otimes \left(\bigoplus_{v'} \mathcal{H}_{v'}\right)$$

$$= \mathcal{H} \otimes \mathcal{H}'$$

$$S(v,v') = k \log \dim \mathcal{H}_{(v,v')}$$

$$= k \log \left(\dim \mathcal{H}_v \dim \mathcal{H}'_{v'}\right)$$

$$= k \log d_v + k \log d'_{v'}$$

$$= S(v) + S(v') \Rightarrow \text{extensive.}$$

$$\mathcal{H} \ni \psi \notin \mathcal{H}_\nu$$

$$\psi = \sum \psi_\nu \,, \qquad \psi_\nu \in \mathcal{H}_\nu$$

entropy operator

$$\tilde{S} = \sum_\nu S(\nu) P_\nu \,, \qquad P_\nu = \text{proj' to } \mathcal{H}_\nu$$

von Neumann 1929

$$\tilde{S}_{vN}(\psi) = -k \sum_\nu \|\psi_\nu\|^2 \log \frac{\overset{1}{\cancel{\|\psi_\nu\|^2}}}{d_\nu} = \sum_\nu \|\psi_\nu\|^2 S(\nu)$$

von Neumann 1927

$$S_{vN}(\rho) = -k \,\mathrm{tr}(\rho \log \rho)$$

S_{vN} is a quantum analog of

$$S_{Gibbs}(\rho) = -k \int \rho \log \rho$$

with $\rho : \Gamma \to [0, \infty)$

Quantum:

$\rho : \mathcal{H} \to \mathcal{H}$, s.a., $\operatorname{tr} \rho = 1$, $\rho \geq 0$,

$$\rho = \sum_n p_n |n\rangle\langle n|,$$

$$-k \operatorname{tr}(\rho \log \rho) = -k \operatorname{tr} \sum_n p_n \log p_n |n\rangle\langle n| = -k \sum_n p_n \log p_n$$

$$= \text{discrete } S_{Gibbs}\left((p_n)_n\right)$$

$$S_{VN} = \log(\text{no. of non-0 eigenvalue})$$

$$\Rightarrow \quad S_{VN}\left(\frac{P_K}{\dim K}\right) = k \log \dim K$$

$$S_{VN}\left(\frac{P_V}{d_V}\right) = k \log d_V = S(V)$$

<u>Conseq</u> $\quad S_{VN}\left(e^{-itH} \rho_0 \, e^{itH}\right) = S_{VN}(\rho_0).$