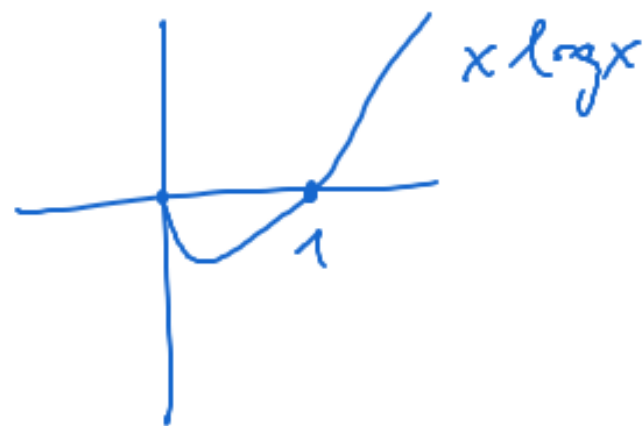


von Neumann entropy

$$S_{vN}(\rho) = -k \operatorname{tr}(\rho \log \rho)$$

$$= -k \sum_n p_n \log p_n$$



$$\rho = \sum_n p_n |u\rangle\langle u|$$

$$S_{vN} = \text{"log (eff. no. of non-0 } p_n \text{"}$$

$$S_{vN}(|\psi\rangle\langle\psi|) = 0, \quad S_{vN}(\rho) \geq 0$$

$$S_{\text{vN}}(\rho) \leq k \log \dim \mathcal{H} = S_{\text{vN}}\left(\frac{I}{\dim \mathcal{H}}\right)$$

$S_{\text{vN}}(\rho)$ = "how mixed" ρ is

Like wise $S_{\text{vN}}(\text{tr}_b(|\psi\rangle\langle\psi|))$ = "how entangled" ψ is.
 $\psi \in \mathcal{S}(\mathcal{H}_a \otimes \mathcal{H}_b)$

Th. eq.: $S(\text{eq}) = k \log d_{\text{eq}} \approx k \log d_{\text{mc}} = S_{\text{vN}}(\rho_{\text{mc}})$
in \mathcal{H}_{mc} $\approx S_{\text{vN}}(\rho_{\text{can}})$
by equiv. of ensembles, $\rho_{\text{can}} \approx \rho_{\text{mc}}$

Then $S_{vN}(\text{tr}_a |\psi\rangle\langle\psi|) = S_{vN}(\text{tr}_b |\psi\rangle\langle\psi|)$

$\psi \in \mathcal{S}(\mathcal{H}_a \otimes \mathcal{H}_b)$

The 3rd Law of Thermodynamics

0th: Every system goes ~~to~~ to th. eq.

1st: Energy is conserved } (R. Clausius
2nd: Entropy increases } 1865)

3rd: (Walther Nernst in 1906)

For every system, the th. eq. entropy
at temperature 0 is $S=0$.

("For every system, the th. eq. entropy at temp. 0
is the same.") wrong in classical mech.

Derivation of 3rd law from QM

- 1) Realistic H are non-degenerate,
- 2) Temperature 0 corresponds to the ground state.

$$T \rightarrow 0 \Leftrightarrow \beta \rightarrow \infty, \quad \rho_{\text{can}} \longrightarrow \rho_{E_0}^{\text{nondeg}} = |\phi_0\rangle\langle\phi_0|$$

$$H = \sum E_\alpha |\phi_\alpha\rangle\langle\phi_\alpha|$$

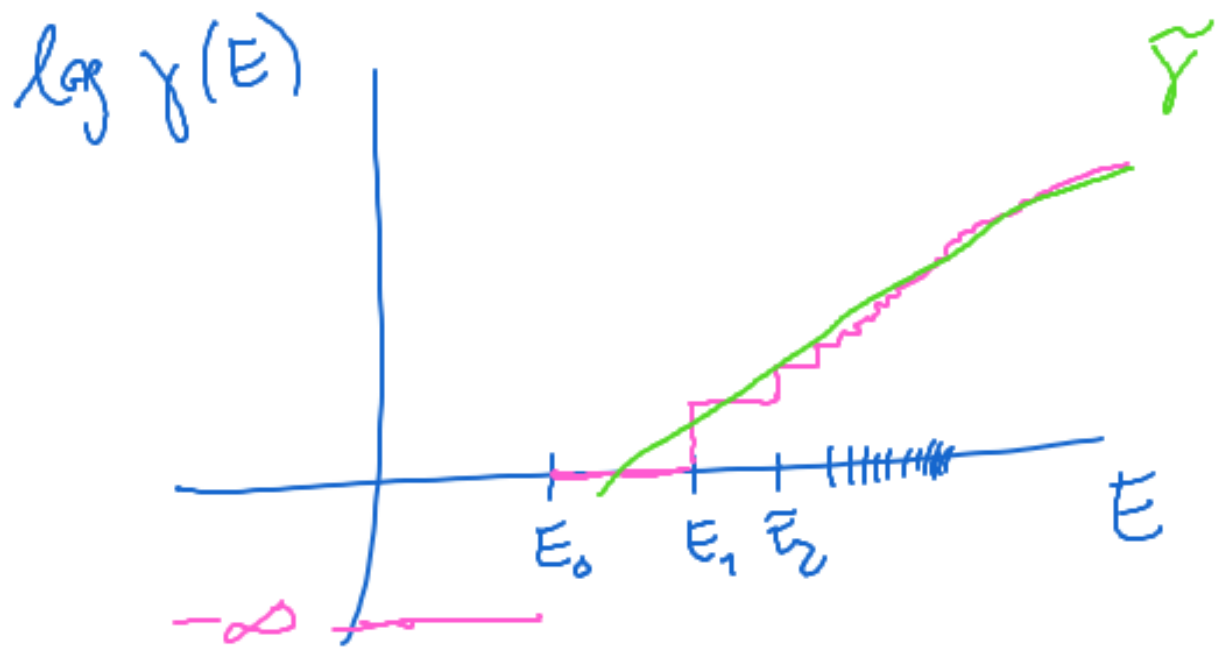
$$\rho_{\text{can}} = \sum_{\alpha} \frac{1}{Z} e^{-\beta E_\alpha} |\phi_\alpha\rangle\langle\phi_\alpha|$$

Or, $S(E, V, N)$ transformed into $S(T, V, N)$

$$E = E(T) \text{ or } E = E(\beta) = \text{tr}(H \rho_{\text{can}}(\beta)) \xrightarrow{\beta \rightarrow \infty} E_0$$

3) The th. eq. entropy is ~~the~~ $k \log \Omega$ in Thuc.

4) It is appropriate to not include excited states in the lowest unc energy interval.



1) - 4) \rightarrow
 $S(T=0, V, N) = 0.$

Macroscopic and Microscopic Thermal Eq.

MATE

MITE

Def of MATE: dominant \mathcal{H}_V , i.e.

$$\frac{\dim \mathcal{H}_{eq}}{\dim \mathcal{H}_{mc}} = 1 - \varepsilon \quad \text{with } \varepsilon \ll 1.$$

Then we say that $\psi \in \delta(\mathcal{H}_{mc})$ is in MATE

iff $\langle \psi | P_{eq} | \psi \rangle \geq 1 - \delta$ with $\varepsilon \ll \delta \ll 1$
 $= \| P_{eq} \psi \|^2$

Prop $u_{mc}(\text{MATE}) \geq 1 - \frac{\varepsilon}{\delta}$ (HW)

(a generalization of the belt theorem)

Rem realistically, $\varepsilon \approx 10^{-10^5}$

Def. of MITE

based on $\rho_s^\Psi \approx \rho_s^{\text{can}}$ for s not too large
and not too strongly
coupled to s^c .

Canonical typicality; for one s .

Now: for all s simultaneously.

Subsystem property: If for s , $\rho_s^\Psi \approx \rho_s^{\text{can}}$

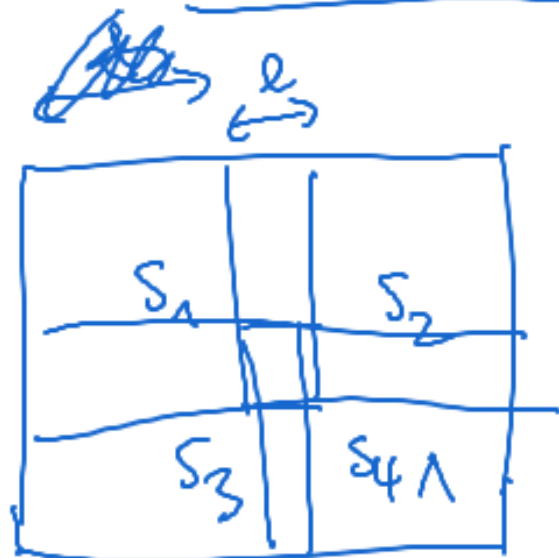
then for $s' < s$, $\rho_{s'}^\Psi \approx \rho_{s'}^{\text{can}}$.

$$\mathcal{H}_s = \mathcal{H}_{s'} \otimes \mathcal{H}_{s|s'}$$

s def'd by a spatial region:



\exists if $\text{diam}(s) \leq l$
 then $\rho_s^\psi \approx \rho_s^{\text{can}}$



⑤ \exists if $\rho_{s_i}^\psi \approx \rho_{s_i}^{\text{can}}$, then

\forall systems s : $\text{diam}(s) \leq l$

$\Rightarrow \rho_s^\psi \approx \rho_s^{\text{can}}$

Know \forall systems with $\text{diam} \leq l$, $\rho_s^\psi \approx \rho_s^{\text{mc}} = \text{tr}_{\text{sc}}(\rho_{\text{mc}})$
 $\approx \rho_s^{\text{can}}$

For $\text{diam}(S) \leq L$, $\rho_{sc}(\rho_{mc}) \approx \rho_{sc}(\rho_{can}) \approx \rho_s^{can}$

Realistically $L \leq l$
 ↑ ↑
 0.1% of 2% of $\text{diam}(\Lambda)$
 $\text{diam}(\Lambda)$

Conseq On length scale L , obtain MITE:

Def MITE = $\left\{ \psi \in \mathcal{S}(\mathcal{H}_{mc}) : \forall \text{subsystems } S \right.$
 $\left. \text{diam}(S) \leq L \Rightarrow \rho_S^\psi \approx \rho_S^{can} \right\}$

Comparison of MITE and MATE

set \mathcal{A} of observables, $\{\psi \in \mathcal{S}(\mathcal{H}_{mc}) : \forall A \in \mathcal{A},$

$$A = \sum a P_a, \forall a: \langle \psi | P_a | \psi \rangle \approx \text{tr}(P_a \rho_{mc})\}$$

"TE _{\mathcal{A}} " . MATE : $\mathcal{A} = \{M_1 \dots M_K\}$

MITE : $\mathcal{A} = \cup \{A_0 \otimes I_{sc} : A_0 \text{ s.a. on } \mathcal{H}_s\}$

$$\langle \psi | P_a | \psi \rangle = \text{tr} \left(\overset{S}{P_{a0}} \rho_s^\psi \right) \approx \text{tr} \left(P_{a0} \rho_s^{\overset{mc}{\psi}} \right)$$

$$P_a = P_{a0} \otimes I_{sc} = \text{tr} (P_a \rho_{mc})$$

MITE \Rightarrow MATE

$$M_j = \sum_s A_s, \quad \psi \in \text{MITE}$$

$$\Rightarrow \text{distr}_\psi(A_s) = \text{dist}_{\rho_{mc}}(A_s)$$

$$\Rightarrow \text{distr}_\psi(M_j) = \text{dist}_{\rho_{mc}}(M_j)$$

$$\Rightarrow \psi \in \text{MATE}, \quad \text{MITE} \subset \text{MATE}.$$

MATE $\not\subset$ MITE: $\psi = \otimes_i \psi_i \notin \text{MITE}$

$$\rho_s^\psi = \text{tr}_{s'} |\psi\rangle\langle\psi| = \sum_{s'=i} |\psi_i\rangle\langle\psi_i| \neq \rho_s^{\text{can}}, \quad \text{typically, } \psi \in \text{MATE}.$$

relevant to thermodynamics: $MATE$.

But in QM, typically (for most ψ),
you have a stronger property: $MITE$.