Operator Algebras

Summer Semester 2023 **Sheet 1**

Exercise 1

(a) Find a Hilbert space $\mathcal{H}$ and bounded operators $A, B \in \mathcal{L}(\mathcal{H})$ such that $AB = \mathrm{Id}$ but neither A nor B are invertible.

(b) Find a submultiplicative matrix norm $\|\cdot\|$ on $\mathbb{C}^{d\times d}$, $d \geq 2$, which is *not* an operator norm, i.e., there does not exist a norm $\|\cdot\|_{\mathbb{C}^d}$ on $\mathbb{C}^d$ such that

$$\|A\| = \sup_{\|v\|_{\mathbb{C}^d}=1} \|Av\|_{\mathbb{C}^d} ,$$

and give a proof of that.

Exercise 2

Show that the Lebesgue integrable functions $L^1(\mathbb{R})$ equipped with the convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y$$

are a commutative Banach algebra (You do not have to show completeness.) Does it have a unit? Prove its (non-)existence.

Exercise 3

Let A be an algebra. An ideal $I \subset A$ is called *proper* if $I \neq A$. A proper ideal I is called *maximal* if, for all proper ideals $J \subset A$, $I \subseteq J$ already implies $I = J$.

Assume now that A is a unital Banach algebra.

(a) Show that the closure of any proper ideal in A is again a proper ideal.

(b) Show that every maximal ideal in A is closed.

(c) Show that every proper ideal in A is contained in a maximal ideal. (Hint: Zorn's lemma)

Exercise 4

Let K be a compact Hausdorff space and let

$$A = C(K) = \{f \colon K \to \mathbb{C} : f \text{ is continuous}\}$$

be equipped with the pointwise multiplication. Show that the maximal ideals in A are exactly of the form $I_x = \{f \in A : f(x) = 0\}$ for some $x \in K$. (Hint: You may use that for any open cover (U_i) of K there exist continuous functions (φ_i) such that $\operatorname{supp}\varphi_i \subseteq U_i$ and $\sum_i \varphi_i = 1$.)

Due Wednesday April 26, 2023 2 pm via postbox in the 3rd floor (Siebert) or URM