Operator Algebras

Summer Semester 2023 **Sheet 10**

Exercise 1 (Bogoliubov transformations)

Let $\mathfrak{h}$ be a Hilbert space.

(a) Let U be a bounded linear operator on $\mathfrak{h}$ and V a bounded antilinear operator on $\mathfrak{h}$ such that

$$V^*U + U^*V = 0 = UV^* + VU^*,$$
$$U^*U + V^*V = \mathrm{Id} = UU^* + VV^*.$$

Show that there exists a unique $*$-automorphism α of $\mathrm{CAR}(\mathfrak{h})$ such that

$$\alpha(a(f)) = a(Uf) + a^*(Vf) \qquad \text{for all } f \in \mathfrak{h}.$$

(b) Let T be an invertible operator on $\mathfrak{h}$ such that $\mathrm{Im}\,\langle Tf, Tg\rangle = \mathrm{Im}\,\langle f, g\rangle$. Show that there exists a unique $*$-automorphism α of $\mathrm{CCR}(\mathfrak{h})$ such that

$$\alpha(W(f)) = W(Tf) \qquad \text{for all } f \in \mathfrak{h}.$$

Exercise 2 (Separability of CAR/CCR)

(a) Show that $\mathrm{CCR}(\mathfrak{h})$ is not separable if $\mathfrak{h} \neq \{0\}$. Hint: Make a proof by contradiction and use that $\|W(f) - 1\| = 2$ for $f \neq 0$.

(b) Show that $\mathrm{CAR}(\mathfrak{h})$ for some inner product space $\mathfrak{h}$ is separable if and only if $\mathfrak{h}$ is separable.

Exercise 3 (Universal C^*-algebras)

(a) Let $X = \{x\}$ and $R = (x = x^*, \|x\| \leq 1)$. Show that $C^*(X|R) \cong C_0([-1,1] \setminus \{0\})$.

(b) Show that, for $\theta \in \mathbb{R}$, the so-called non-commutative torus $A_\theta = C^*(\{u, v\}|uu^* = u^*u = 1,\ uv = e^{2i\pi\theta}vu\}$ exists and is non-trivial. Hint: Consider on $\mathcal{L}(L^2(\mathbb{S}^1))$ the operators

$$U(f)(z) = zf(z), \qquad V(f)(z) = f(ze^{-2\pi i\theta}).$$

Exercise 4 (Fock representation is regular)

Let $\mathfrak{h} \neq \{0\}$ be an inner product space.

(a) Show that, for $f \neq 0$, $\mathbb{R} \to \mathrm{CCR}(\mathfrak{h})$, $t \mapsto W(tf)$ is not continuous with respect to the norm topology in $\mathrm{CCR}(\mathfrak{h})$.

(b) Show that, for all $f \in \mathfrak{h}$, $\mathbb{R} \to \mathrm{CCR}_F(\mathfrak{h})$, $t \mapsto W(tf)$ is continuous with respect to the strong operator topology.

Due Wednesday **July 12, 2023** in the lecture or via e-mail