Operator Algebras

Summer Semester 2023 **Sheet 11**

Exercise 1 (Multiplication in operator topologies I)

(a) Let $A \in \mathcal{L}(\mathcal{H})$. Show that multiplication by A (as a map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$) from left and right is continuous in the weak and strong operator topology.

(b) Conclude from (a): Let $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$. Then the commutant S' is closed in the weak operator topology (and therefore also in the strong operator and norm topology).

Exercise 2 (Joint continuity of multiplication and adjoints in operator topologies)

(a) The multiplication map $\cdot\colon \mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}), (A, B) \mapsto AB$ is in general not strongly continuous, but it is strongly continuous on bounded subsets of $\mathcal{L}(\mathcal{H})$.

(b) The adjoint map $*\colon \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, $A \mapsto A^*$, is continuous in the weak operator topology but not in the strong operator topology.

Exercise 3 (Bases for operator topologies)

(a) For $A_1, \dots, A_n \in \mathcal{L}(\mathcal{H})$, $\xi_1, \dots, \xi_n \in \mathcal{H}$, let

$$N_s((A)_{i=1}^n, (\xi)_{i=1}^n) := \{B \in \mathcal{L}(\mathcal{H}) : \forall i = 1, \dots, n : \|(B - A_i)\xi_i\| < 1\},$$

and let $\mathcal{N}_s := \{N_s((A)_{i=1}^n, (\xi)_{i=1}^n) : n \in \mathbb{N},\ A_1, \dots, A_n \in \mathcal{L}(\mathcal{H}),\ \xi_1, \dots, \xi_n \in \mathcal{H}\}$. Show that $\mathcal{N}_s$ is a topological basis for the strong operator topology defined as in the lecture of the seminorms, i.e., show for that for every open set $U \subseteq \mathcal{L}(H)$ and $A \in U$ there is an element $N \in \mathcal{N}_s$ such that $A \in N \subseteq U$.

(b) For $A_1, \dots, A_n \in \mathcal{L}(\mathcal{H})$, $\eta_1, \dots, \eta_n, \xi_1, \dots, \xi_n \in \mathcal{H}$, let

$$N_w((A)_{i=1}^n, (\eta)_{i=1}^n, (\xi)_{i=1}^n) := \{B \in \mathcal{L}(\mathcal{H}) : \forall i = 1, \dots, n : |\langle \eta_i, (B - A_i)\xi_i \rangle| < 1\},$$

and let $\mathcal{N}_w = \{N_w((A)_{i=1}^n, (\xi)_{i=1}^n) : n \in \mathbb{N},\ A_1, \dots, A_n \in \mathcal{L}(\mathcal{H}), \eta_1, \dots, \eta_n, \xi_1, \dots, \xi_n \in \mathcal{H}\}$. Show that $\mathcal{N}_w$ is a topological basis for the weak operator topology.

Exercise 4 (A commutative von Neumann algebra)

Let (X, μ) be a finite measure space. Show that the space of essentially bounded functions $L^\infty(X) \subseteq \mathcal{L}(L^2(X, \mu))$ is a von Neumann algebra, where the inclusion is to be understood in the sense that each function $f \in L^\infty(X)$ can be understood as a (bounded) multiplication operator $M_f \in \mathcal{L}(L^2(X, \mu))$. Hint: It suffices to show $L^\infty(X) = L^\infty(X)''$. To this end consider for $A \in L^\infty(X)'$, the function $f = A(1_X) \in L^2(X, \mu)$. Show that $f \in L^\infty(X)$ and $A = M_f$.

Due Wednesday **July 19, 2023** in the lecture or via e-mail