Operator Algebras

Summer Semester 2023 **Sheet 2**

Exercise 1

Let A be a Banach algebra and $a, b \in A$. Show that

(a) $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$,

(b) $\sigma(ab) = \sigma(ba)$ is *false* in general,

(c) $ab, ba \in \operatorname{Inv}(A) \implies a, b \in \operatorname{Inv}(A)$.

Exercise 2

Let $D := B_1(0) = \{z \in \mathbb{C} : |z| \leq 1\}$ be the complex disc and let $U := D^\circ = \{z \in \mathbb{C} : |z| < 1\}$ be its inner part. Consider

$$A = \{f \in C(D) : f \text{ is holomorphic on } U\}$$

equipped with $\|f\|_\infty := \sup_{z \in D} |f(z)|$. Show that A with the pointwise multiplication is a Banach algebra and $\sigma_A(f) = f(D)$ for all $f \in A$.

Exercise 3 (Spectrum depends on the algebra)

Let A be as in Exercise 1 and $\mathbb{S} := \{z \in \mathbb{C} : |z| = 1\}$. Consider $C(\mathbb{S})$ with $\|\cdot\|_\infty$ as Banach algebra. Show the following.

(a) The restriction map $\phi \colon A \to C(\mathbb{S})$, $f \mapsto f\restriction_{\mathbb{S}}$ is an isometric algebra homomorphism.

(b) $B := \phi(A) \subseteq C(\mathbb{S})$ can be considered as subalgebra of $C(\mathbb{S})$ and

$$\sigma_B(\phi(f)) = f(D), \qquad \sigma_{C(\mathbb{S})}(\phi(f)) = f(\mathbb{S}).$$

(c) Find $g \in B$ such that $\sigma_B(g) \neq \sigma_{C(\mathbb{S})}(g)$.

Exercise 4

Consider $\ell^1(\mathbb{Z})$ with the convolution as algebra (as in Example 1.2 (6) of the lecture).

(a) Show explicitly that it is a commutative Banach algebra (take again for granted that it is complete).

(b) Show that

$$\phi : \ell^1(\mathbb{Z}) \to C(\mathbb{S}),\ f \mapsto \widehat{f}, \qquad \widehat{f}(z) := \sum_{n \in \mathbb{Z}} f(n) z^n$$

is a well-defined continuous algebra homomorphism.

(c) Conclude that $\widehat{f}(\mathbb{S}) \subseteq \sigma_{\ell^1(\mathbb{Z})}(f)$.

Due Wednesday May 3, 2023 2 pm via postbox in the 3rd floor (Siebert) or URM