Operator Algebras

Summer Semester 2023 **Sheet 3**

Exercise 1

Show that each proper ideal in the algebra of the matrices $\mathbb{C}^{d\times d}$ is of the form $\{0\}$.

Exercise 2

Let X be a compact Hausdorff space. Show that the map

$$\delta\colon X \to \widehat{C(X)}, \quad x \mapsto \delta_x,$$

with $\delta_x(f) := f(x)$, is a homeomorphism. Conclude that the Geldfand transformation acts under this isomorphism as the identity on $C(X)$.

Exercise 3

Consider $\ell^1(\mathbb{Z})$ with the convolution and $\|\cdot\|_1$ as Banach algebra.

(a) Show that each $z \in \mathbb{S} \subseteq \mathbb{C}$ defines an element $\chi_z \in \widehat{\ell^1(\mathbb{Z})}$ given by

$$\chi_z(f) := \widehat{f}(z) := \sum_{n\in\mathbb{Z}} f(n)z^n,$$

and the map $\mathbb{S} \to \widehat{\ell^1(\mathbb{Z})}$, $z \mapsto \chi_z$ is a homeomorphism.

(b) Prove that $f \in \ell^1(\mathbb{Z})$ is invertible if and only if $\widehat{f}(z) \neq 0$ for all $z \in \mathbb{S}$.

(c) Prove that $\sigma(f) = \widehat{f}(\mathbb{S})$ for all $f \in \ell^1(\mathbb{Z})$.

Exercise 4

Let A be a C^*-algebra.

(a) Let $a, b, u \in A$ such that u is unitary and $b = u^*au$. Show that $\sigma(a) = \sigma(b)$.

(b) Let $p \in A$ be a projection, i.e., $p^2 = p$ and $p^* = p$ and assume it is non-trivial ($p \notin \{0, 1_A\}$). Compute $\sigma(p)$.

Due Wednesday May 10, 2022 in the lecture or via e-mail