Operator Algebras

Summer Semester 2023 **Sheet 4**

Exercise 1

The common real version of the Stone-Weierstraß theorem states: Let X be a compact Hausdorff space and $\mathcal{A}$ be a subalgebra of the real-valued continuous functions $C(X, \mathbb{R})$ which contains a non-zero constant function. Then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$ if and only if it separates points.

Show that this version implies the complex version for $C_0(X)$ on locally compact spaces X, which was used in the lecture for proving the Gelfand-Naimark theorem.

Exercise 2

Let A be a C^*-algebra and let $f(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ (with $\alpha_0 = 0$ if A is not unital) be a power series with $R > 0$ being the radius of convergence. Let $a \in A$ with $\sigma(a) \subseteq B_R(z_0)^\circ = \{z \in \mathbb{C} : |z - z_0| < R\}$. Show that

$$f(a) := \Phi_a(f) = \sum_{n=0}^{\infty} \alpha_n (a - z_0)^n.$$

Exercise 3

Let A be a C^*-algebra and $a \in A$ normal. Show that

(a) $\|(\lambda - a)^{-1}\| = \operatorname{dist}(\lambda, \sigma(a))^{-1}$,

(b) for all $\epsilon > 0$ and all $b \in A$ with $\|a - b\| < \epsilon$, one has

$$\sigma(y) \subseteq \{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, \sigma(a)) < \epsilon\}.$$

Exercise 4

Let A be a C^*-algebra. Show the following.

(a) Let $u \in A$ be normal. Then u is unitary if and only if $\sigma(u) \subseteq \mathbb{S}$.

(b) If $a \in A$ is self-adjoint, then $u = e^{ia}$ is unitary.

(c) If $u \in A$ is unitary and $\sigma(u) \neq \mathbb{S}$, then there exists a self-adjoint $a \in A$ such that $u = e^{ia}$.

Due Wednesday May 17, 2022 in the lecture or via e-mail