Operator Algebras

Summer Semester 2023 **Sheet 5**

Exercise 1 (Finishing the proof of the positivity characterization theorem of the lecture)

Let A be a C^*-algebra and $a \in A^+$. Assume $c \in A^+$ with $ca = ac$ and $c^2 = a$. Show that $c = \sqrt{a}$. Hint: Consider the case $A = C_0(X)$ first.

Exercise 2

Let A be $*$-algebra. An element $v \in A$ is called a partial isometry if v^*v is a projection.

(a) Show that v is a partial isometry if and only if $v = vv^*v$. Show that in this case vv^* is a projection as well.

(b) Let $A = \mathbb{C}^{d\times d}$ and let $p, q \in A$ be projections. If $\operatorname{tr} p \leq \operatorname{tr} q$, then there exists a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* \leq q$.

(c) Two projections $p, q \in A$ are called Murray-von-Neumann equivalent if there exits a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$. Verify that it is an equivalence relation and describe the equivalence classes in the case of $A = \mathbb{C}^{d\times d}$.

Exercise 3

Let A be a separable C^*-algebra. Show that there exists a sequential approximate unit with the same properties as proven in the lecture, i.e., we can choose $\Lambda = \mathbb{N}$.

Exercise 4

Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $(u_n)_{n\in\mathbb{N}}$. Let $\Lambda := \{\lambda \subset \mathbb{N} : \lambda \text{ is finite}\}$ ordered by inclusion as in the existence proof in the lecture. For $\lambda \in \Lambda$ let $e_\lambda := \sum_{k\in\lambda} \langle u_k, \cdot\rangle\, u_k$. Show that $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for the C^*-algebra of compact operators $\mathcal{K}(\mathcal{H})$.

Due Wednesday May 24, 2022 in the lecture or via e-mail