Operator Algebras

Summer Semester 2023 **Sheet 6**

Exercise 1 (Commutative quotient C^*-algebras I)

Let A be a commutative C^*-algebra, $I \subseteq A$ a closed ideal and let $q\colon A \to A/I$ be the canonical quotient map. Show the following.

(a) The set $\widehat{A}_I := \{\chi \in \widehat{A} : \chi|_I = 0\}$ is a closed subset of $\widehat{A}$.

(b) The map $q^*\colon \widehat{A/I} \to \widehat{A}_I$, $\chi \mapsto \chi \circ q$ is a well-defined homeomorphism.

(c) The map $\widehat{A} \setminus \widehat{A}_I \to \widehat{I}$, $\chi \mapsto \chi|_I$ is a homeomorphism. Hint: First show that each $\chi \in \widehat{I}$ has a unique extension $\widetilde{\chi} \in \widehat{A}$ by considering the equation $\widetilde{\chi}(ab) = \widetilde{\chi}(a)\chi(b)$ for $a \in A,\ b \in I$.

Exercise 2 (Commutative quotient C^*-algebras II)

Let X be a locally compact Hausdorff space. Show the following.

(a) For every closed subset $Y \subseteq X$ the set $I_Y := \{f \in C_0(X) : f|_Y = 0\}$ is a closed ideal in $C_0(X)$, and $C_0(X)/I_Y \cong C_0(Y)$.

(b) For every closed subset $Y \subseteq X$ we have $I_Y \cong C_0(X \setminus Y)$.

(c) For every closed ideal $I \subseteq C_0(X)$ there exists a unique closed subset $Y \subseteq X$ such that $I = I_Y$.

Hint: In (a), you may use Urysohn's lemma, the Tietze extension theorem and the representation of $C_0(X)$ via the one-point compactification X_∞. In (b) and (c) Exercise 1 can be useful.

Exercise 3 (Vector states)

Let A be a C^*-algebra, $\pi\colon A \to \mathcal{L}(\mathcal{H})$ be a representation on some Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$. Consider the functional

$$\omega_{\pi,\Omega}(a) := \langle \Omega, \pi(a)\Omega \rangle\,, \qquad a \in A.$$

(a) Show that $\omega_{\pi,\Omega}$ defines a positive linear functional with $\|\omega_{\pi,\Omega}\| \leq \|\Omega\|^2$.

(b) If π is non-degenerate, prove that $\pi(e_\lambda)\Omega \to \Omega$ for any approximate unit (e_λ) with $\|e_\lambda\| \leq 1$.

(c) If π is non-degenerate, then $\|\omega_{\pi,\Omega}\| = \|\Omega\|^2$ and, in particular, $\omega_{\pi,\Omega}$ defines a state if and only if $\|\Omega\| = 1$.

Exercise 4 (Faithful representations in finite dimensions)

Let $A = \mathbb{C}^{n\times n}$ and $\mathcal{H} = \mathbb{C}^m$. For which values m there exists a faithful representation of A on $\mathcal{H}$? Find two faithful representations π_1, π_2 of A on $\mathbb{C}^m$ for some m such that π_1 and π_2 are not equivalent.

Due Wednesday June 07, 2022 **after the semester break** in the lecture or via e-mail