Operator Algebras

Summer Semester 2023 **Sheet 7**

Exercise 1 (GNS representation in finite dimensions)

Let ρ be a density matrix on $\mathbb{C}^d$, that is, $\rho \in \mathbb{C}^{d\times d}$ with $0 \leq \rho \leq 1$ and $\operatorname{tr}\rho = 1$. Consider the state $\omega_\rho \in S(\mathbb{C}^{d\times d})$ given by

$$\omega_\rho(a) := \operatorname{tr}(\rho a).$$

(a) Assume that all eigenvalues of ρ are strictly positive. Describe explicitly the GNS representation $(\mathcal{H}, \pi, \Omega)$ with respect to ω_ρ, i.e., give a concrete choice for $(\mathcal{H}, \pi, \Omega)$.

(b) What happens if some eigenvalues of ρ are zero?

Exercise 2 (Irreducible representations and states on commutative algebras)

Let A be a C^*-algebra and let $\pi\colon A \to \mathcal{L}(\mathcal{H})$ be an irreducible representation of A. Show the following.

(a) If $b \in A'$, i.e., $ab = ba$ for all $a \in A$, then there exists $z \in \mathbb{C}$ such that $\pi(b) = z \operatorname{Id}_{\mathcal{H}}$.

Assume now that A is commutative.

(b) We have $\mathcal{H} \cong \mathbb{C}$ and π is equivalent to a $*$-homomorphism $\chi\colon A \to \mathbb{C} \cong \mathcal{L}(\mathbb{C})$.

(c) The pure states on A are exactly given by the characters $\widehat{A}$.

Exercise 3 (States on $C(X)$)

Let X be a compact Hausdorff space. According to a version of Riesz' theorem there is a bijection between finite positive Borel measures μ on X and the positive linear functionals on $C(X)$, which is given by

$$\mu \mapsto \phi_\mu, \qquad \text{where } \phi_\mu(f) := \int_X f(x)\mathsf{d}\mu(x).$$

(a) Show that this bijection also yields a bijection between the space of all (Borel) probability measures on X and $S(C(X))$, the set of states of $C(X)$.

(b) Let μ be a probability measure, ϕ_μ defined as above and $(\mathcal{H}_{\phi_\mu}, \pi_{\phi_\mu}, \Omega_{\phi_\mu})$ be the GNS representation with respect to ϕ_μ. Show that there exists a unitary operator $U\colon \mathcal{H}_{\phi_\mu} \to L^2(X, \mu)$ such that, for all $f \in C(X)$,

$$U\pi_{\phi_\mu}(f) = M_f U, \qquad U\Omega_{\phi_\mu} = 1_X,$$

where $M_f(g) := f \cdot g$, $g \in L^2(X, \mu)$, denotes the multiplication operator by f on $L^2(X, \mu)$ and $1_X \in L^2(X, \mu)$ the constant function with value one.

(c) Which probability measures correspond to the pure states on $C(X)$?

Hint: You can use that $C(X)$ is dense in $L^2(X, \mu)$.

Due Wednesday June 14, 2022 in the lecture or via e-mail