Operator Algebras

Summer Semester 2023 **Sheet 8**

Exercise 1 (The algebra of matrices with entries in a C^*-algebra)

Let A be a C^*-algebra. For $d \in \mathbb{N}$, let $A^{d\times d}$ be the $d \times d$ matrices with entries in A with the pointwise addition, canonical matrix multiplication and involution, i.e. for $B, C \in A^{d\times d}$, $(BC)_{ij} := \sum_k B_{ik}C_{kj}$ and $(B^*)_{ij} := (B_{ji})^*$. It is clear that $A^{d\times d}$ is a $*$-algebra. Show that it also admits a (unique) C^*-norm and therefore becomes a C^*-algebra as well. Hints:

(1) First assume that A is a concrete C^*-algebra, i.e. $A \subseteq \mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then we have
$$A^{d\times d} \subseteq \mathcal{L}\left(\oplus_{l=1}^{d}\mathcal{H}\right)$$
and can use the appropriate norm in $\mathcal{L}\left(\oplus_{l=1}^{d}\mathcal{H}\right)$.

(2) Reduce the general abstract case to the concrete case by using a famous theorem of the lecture.

Exercise 2 (Irreducible representations of matrix algebras)

Let $A = \mathbb{C}^{d\times d}$ for some $d \in \mathbb{N}$. Show that:

(a) The canonical representation $\mathrm{Id}\colon A \to A \cong \mathcal{L}(\mathbb{C}^d)$ is irreducible.

(b) Every irreducible representation π of A is equivalent to the one in (a). Hint: First show that $\pi(\delta_{11}) \neq 0$ for every irreducible representation π. Then choose $v_1 = \pi(\delta_{11})\xi$ for some $\xi \in \mathbb{C}^d$ such that $\|v_1\| = 1$ and, for each $i = 1, \dots, d$, consider $v_i = \pi(\delta_{i1})v_1 = \pi(\delta_{i1})\xi$ and show that $(v_1, \dots, v_n)$ behaves under π as the standard basis under the identity representation.

(c) $S_p(A)$ equipped with the weak-$*$-topology is homeomorphic to $\mathbb{S}^{2d-1}/\sim$, where $\mathbb{S}^{2d-1} := \{z \in \mathbb{C}^{2d} : \|z\| = 1\}$ and $v \sim w :\Longleftrightarrow \exists z \in \mathbb{S}^1 \subseteq \mathbb{Z} : zv = w$ for $v, w \in \mathbb{S}^{2d-1}$.

Exercise 3 (Extensions of representations)

Let A be a C^*-algebra and $\{0\} \neq I \subseteq A$ a closed ideal in A. Show that if $\pi\colon I \to \mathcal{L}(\mathcal{H})$ is a non-degenerate representation of I, then there exists a unique representation $\widetilde{\pi}\colon A \to \mathcal{L}(\mathcal{H})$ of A such that $\widetilde{\pi}|_I = \pi$. Hint: For every $\xi \in \mathcal{H}_0 := \pi(I)\mathcal{H}$, we find $\widetilde{\pi}(a)\xi = \lim_\lambda \pi(ae_\lambda)\xi$ for a suitable approximate unit (e_λ) in I.

Due Wednesday June 21, 2022 in the lecture or via e-mail