Operator Algebras

Summer Semester 2023 **Sheet 9**

Exercise 1 (Tensor product polarization formula)

Let E be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Show that for $v_1, \ldots, v_n \in E$,

$$\begin{aligned} n! P_{+,n}(v_1 \otimes \ldots \otimes v_n) &:= \sum_{\tau \in S_n} v_{\tau(1)} \otimes \ldots \otimes v_{\tau(n)} \\ &= \frac{1}{2^{n-1}} \sum_{\sigma_2, \ldots, \sigma_n \in \{\pm 1\}} \sigma_2 \cdots \sigma_n (v_1 + \sigma_2 v_2 + \ldots + \sigma_n v_n)^{\otimes n}. \end{aligned}$$

Hint: One possibility is to first show that

$$\sum_{\tau \in S_n} v_{\tau(1)} \otimes \ldots \otimes v_{\tau(n)} = \int_\Omega X_1(\omega) \cdots X_n(\omega) \left(\sum_{j=1}^n X_j(\omega) v_k \right)^{\otimes n} \mathsf{d}\mathbb{P},$$

where $(\Omega, \mathbb{P})$ is a probability space, $X_1, \ldots, X_n$ stochastically independent real-valued random variables, which are normalized in the L^2 sense and centered, i.e., $\int_\Omega X_j \mathsf{d}\mathbb{P} = 0$ and $\int_\Omega X_j^2 \mathsf{d}\mathbb{P} = 1$. Then consider the concrete situation $\Omega = \{-1, 1\}^{\mathbb{N}}$, $\mathbb{P}$ being product measure of $\frac{1}{2}(\delta_{-1} + \delta_1)$ and $X_j((x_n)_{n \in \mathbb{N}}) := x_j$.

Exercise 2 (Finite-dimensional CAR algebra)

Let E be a finite-dimensional vector space with orthonormal basis $f_1, \ldots, f_n$. Show that $(\mathbb{C}^{2\times 2})^{\otimes n}$ (with component-wise multiplication) is an (abstract) CAR algebra over E, where (some of) the generators are given by

$$a(f_k) = \prod_{j=1}^{k-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{(j)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_{(k)},$$

with $A_{(j)} := \mathrm{Id} \otimes \cdots \otimes \underbrace{A}_{j\text{-th position}} \otimes \cdots \otimes \mathrm{Id} \in (\mathbb{C}^{2\times 2})^{\otimes n}$ for $A \in \mathbb{C}^{2\times 2}$. Hint: In order to show that the $a(f)$, $v \in E$, actually generate the whole algebra show that

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}_{(k)} =& \alpha a(f_k) a^*(f_k) + \beta \prod_{j=1}^{k-1} \big(1 - 2a^*(f_j) a(f_j)\big) a(f_k) \\ &+ \gamma \prod_{j=1}^{k-1} \big(1 - 2a^*(f_j) a(f_j)\big) a^*(f_k) + \delta a^*(f_k) a(f_k) \end{aligned}$$

for all $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

Exercise 3 (Fock representation is cyclic and irreducible)

Consider the canonical inclusion $\iota\colon \mathrm{CCR}_F(\mathfrak{h}) \to \mathcal{L}(\mathfrak{F}_+(\mathfrak{h}))$ as representation. Show the following

(a) The vacuum Ω is a cyclic vector for ι.

(b) For $f, g \in \mathfrak{h}$,

$$\lim_{t\to 0} \frac{W(itf) - \mathrm{Id}}{it}\epsilon(g) = \phi(f)\epsilon(g), \qquad \phi(f) := a_+^*(f) + a_+(f).$$

This implies that $\lim_{t\to 0} \frac{W(itf)-\mathrm{Id}}{it}\psi = \phi(f)\psi$ for all $\psi \in \mathfrak{F}_+(\mathfrak{h}) \cap \mathfrak{F}_{\mathrm{fin}}(\mathfrak{h})$.

(c) Any bounded operator $T \in \mathrm{CCR}_F(\mathfrak{h})'$ satisfies $T\Omega = z\Omega$ for some $z \in \mathbb{C}$, and

$$Ta_+^*(f_1)\cdots a_+^*(f_n)\Omega = a_+^*(f_1)\cdots a_+^*(f_n)T\Omega.$$

Hint: Use (b) and proceed as in the proof for the CAR. Use that $a_+(f)$ and $a_+^*(f)$ can be written as a linear combination of $\phi(g)$, $g \in \mathfrak{h}$.

(d) The representation ι is irreducible.

Due Wednesday **July 05, 2023** in the lecture or via e-mail