Quantum Information Theory **Universität Tübingen**
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Basic Notions in Quantum Computation

Warm-up exercises

Problem 1. Bra-kets and inner products

1. Write down the matrix representation for the following expressions:
   a) $|0\rangle\langle 1|$.
   b) $|0\rangle\langle 0| + |1\rangle\langle 1|$.
   c) $|+\rangle\langle 0|$.
2. Define the Hadamard gate as
$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$
   Express it in the computational basis.
3. Compute $H|+\rangle$ using the bra-ket notation.
4. Let $|\varphi\rangle, |\psi\rangle \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n\times n}$. Prove that the following holds:
$$\langle\varphi|A\psi\rangle = \langle A^*\varphi|\psi\rangle .$$
5. Prove that unitary matrices are norm-preserving.

Problem 2. Tensor products.

1. Consider two vectors $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{C}^2$. Express the product
$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$
   as a vector in $\mathbb{C}^4$.
2. Given two matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2}$, write an expression for
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$
   as a matrix of dimension 4×4.

Graded exercises

Problem 3. Bell states. Let us define the Bell states as

$$
\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right), \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right).
\end{aligned}
$$

Show that they form an orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Problem 4. Pauli matrices. The Pauli matrices are given by:

$$
I = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

1. Verify the following commutation relations:

$$
[X, Y] = 2iZ, \quad [Y, Z] = 2iX, \quad [Z, X] = 2iY.
$$

2. Verify the following anticommutation relations:

$$
A, B = 2\delta_{A,B}, \quad \text{for } A, B \in \{X, Y, Z\}.
$$

3. Compute the eigenvalues of the Pauli matrices.

Problem 5. Hermitian matrices of dimension 2×2. Let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$. Consider the following matrix:

$$
A = \begin{pmatrix} \frac{1+\alpha}{2} & \beta \\ \beta^* & \frac{1-\alpha}{2} \end{pmatrix}.
$$

1. Show that A is Hermitian and $\mathrm{tr}(A) = 1$.
2. Show that any Hermitian matrix in dimension 2×2 with unit trace can be expressed in this form.
3. Show that A can be uniquely expressed as a linear combination of the Pauli matrices.

Now, consider an arbitrary 2×2 Hermitian matrix ρ with unit trace, which we can write as

$$
\rho = \frac{1}{2}(I + n_x X + n_y Y + n_z Z)
$$

with $n_x, n_y, n_z \in \mathbb{R}$.

1. Compute the eigenvalues of ρ and express them in terms of n_x, n_y, n_z.
2. Denote $\mathbf{n} := (n_x, n_y, n_z)$. Show that $\rho \geq 0$ if, and only if, $|\mathbf{n}| \leq 1$.
3. Show that if $|\mathbf{n}| = 1$, then there are $v_1, v_2 \in \mathbb{C}$ such that

$$
\rho = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} v_1^* & v_2^* \end{pmatrix}.
$$

Challenge exercise

Problem 6. Distinguishing quantum states

1. Prove that no measurement can distinguish two non-orthogonal states with probability 1.
2. Prove that, given a qubit in either of two non-orthogonal states $|\varphi_1\rangle, |\varphi_2\rangle$, there is a POVM (E_1, E_2, E_3) with three outcomes $1, 2, 3$ such that if the outcome is 1, then the state is $|\varphi_1\rangle$, if the outcome is 2, then the state is $|\varphi_2\rangle$ and nothing can be concluded if the outcome is 3.