Quantum Information Theory **Universität Tübingen**
SoSe 2023 Ángela Capel
Paul Gondolf

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Quantum entanglement and quantum gates

Warm-up exercises

Problem 1. Let $A, B \in \mathbb{C}^{n\times n}$ be two positive semi-definite matrices.

1. Show that for any $0 \leq t$, $A + tB$ is positive semi-definite.
2. Consider the matrix $M(t) := A + tB$ for $t > 0$. Show that there is a $t > 0$ such that $M(t) \geq 0$ if, and only if, $\mathrm{Ker}(A) \subseteq \mathrm{Ker}(B)$, where $\mathrm{Ker}(M) := \{x \in \mathbb{C}^n : Mx = 0\}$.
3. Let us recall that the exponential of a matrix $A \in \mathbb{C}^{n\times n}$ is defined as

$$\exp(A) := \sum_{j=0}^{\infty} \frac{A^j}{j!} \,.$$

   Prove that, for any unitary matrix $U \in \mathbb{C}^{n\times n}$, the following holds:

$$\exp(UAU^*) = U \exp(A)\, U^* \,.$$

Problem 2.

1. Show that
   a) $HXH = Z\,,$
   b) $HZH = X,$

   where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad , \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

2. Prove for every $\theta \in \mathbb{R}$ the following:

$$HR_x(\theta)H = R_z(\theta)\,,$$

   where

$$R_x(\theta) := \mathrm{e}^{-i\theta X/2}\,,$$
$$R_z(\theta) := \mathrm{e}^{-i\theta Z/2}\;.$$

Graded exercises

Problem 3. Let us consider a qubit, which undergoes three different scenarios:

1. We measure the qubit in the X basis, and the outcome of the measurement is reported.
2. We measure the qubit in the Z basis, and the experiment continues only if the outcome is 1. If so, we measure the qubit in the X basis, and the outcome of the measurement is reported.
3. We measure the qubit in the Z basis first, and the outcome of the measurement is discarded. Next, we measure the qubit with X, and the outcome of the measurement is reported.

Consider the initial qubit to be

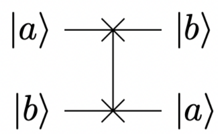
1. A pure state $|\varphi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}\,|1\rangle)$.
2. A mixed state $\rho = \frac{1}{3}(|0\rangle\langle 0| + 2\,|1\rangle\langle 1|)$.

Compute the probability of the measurement outcomes in the first and third scenarios for both possibilities of initial qubits. Next, compare the second and third scenarios to justify why discarding the measurement or observing it makes a difference in the subsequent measurement outcomes.

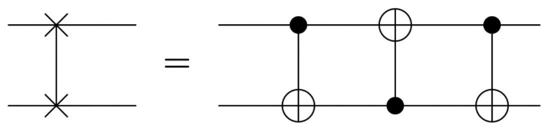
Hint: Measuring in the X basis means measuring with $\{|+\rangle, |-\rangle\}$, whereas measuring in the Z basis means measuring with $\{|0\rangle, |1\rangle\}$.

Problem 4.

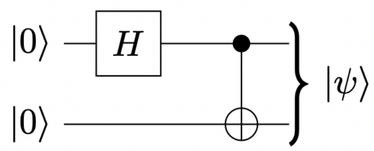
1. Express in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ the swap-gate, which maps $|a, b\rangle \mapsto |b, a\rangle$ and in circuit form is written as



2. Show that the swap-gate operation is equivalent to three CNOT gates, which is represented as



3. Compute the output $|\psi\rangle$ of the following circuit



4. Construct the CNOT gate from the controlled-Z gate and two Hadamard gates.

Problem 5. Show that unitaries cannot delete information, i.e., that there does not exist a 1-qubit unitary U that maps

$$|\phi\rangle \mapsto |0\rangle$$

for every 1-qubit state $|\phi\rangle$.

Challenge exercise

Problem 6. Assume that Alice and Bob share an entangled state ρ_{AB}, ideally the Bell state:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) .$$

Additionally, Alice has a single qubit state $|\phi_A\rangle$, which she would like to teleport to Bob. Let ρ_B the state received by Bob after the teleportation protocol. We can define the fidelity of this protocol as

$$F := \min_{|\phi_A\rangle} \langle\phi_A|\rho_B|\phi_A\rangle \ .$$

Compute this fidelity in the following two cases:

1. $\rho_{AB} = (1-\varepsilon)\,|\Phi^+\rangle\,\langle\Phi^+| + \varepsilon\frac{I}{4}$, where $\varepsilon \in (0,1)$.
2. $\rho_{AB} = |\Phi(\varepsilon)\rangle\,\langle\Phi(\varepsilon)|$, where $|\Phi(\varepsilon)\rangle = \sqrt{1-\varepsilon}\,|00\rangle + \sqrt{\varepsilon}\,|11\rangle$ and $\varepsilon \in (0,1)$.

Sketch these fidelities as functions of ε.