Quantum Information Theory **Universität Tübingen**
SoSe 2023

Ángela Capel
Paul Gondolf

Sheet 6 14. June 2023

Quantum channels and open systems

Graded exercises

Problem 1. Consider the following two-qubit Hamiltonian:

$$H(\Delta) = Z \otimes Z + \Delta(X \otimes I + I \otimes X)\,.$$

We want to study the properties of the thermal state ρ_β:

$$\rho(\Delta, \beta) = \frac{\mathrm{e}^{-\beta H(\Delta)}}{\mathrm{Tr}[\mathrm{e}^{-\beta H(\Delta)}]}$$

as a function of Δ and β.

1. Compute the eigenvalues of $H(\Delta)$.
2. Consider $\rho(\beta \to 0, \Delta)$. For which values of Δ is the thermal state separable, and for which entangled?
3. Consider $\rho(\beta \to \infty, \Delta)$. For which values of Δ is the thermal state separable, and for which entangled?

(Hint: In 1. it might be easier to study how each of the terms of the Hamiltonian act on Bell states)

Problem 2. Consider a quantum channel $\mathcal{E}$ on a qubit which measures the operator X on the qubit and:

- If the outcome of the measurement is $+1$, the qubit is erased and replaced with the state $\frac{1+\varepsilon_+}{2}\left|0\right\rangle\left\langle 0\right| + \frac{1-\varepsilon_+}{2}\left|1\right\rangle\left\langle 1\right|$.
- If the outcome of the measurement is $+1$, the qubit is erased and replaced with the state $\frac{1+\varepsilon_-}{2}\left|0\right\rangle\left\langle 0\right| + \frac{1-\varepsilon_-}{2}\left|1\right\rangle\left\langle 1\right|$.

In both cases, $\varepsilon_\pm \in (-1, 1)$.

1. Write an expression for $\mathcal{E}(\rho)$ in terms of ρ, where ρ is the single qubit state on which the map is applied. Show that the map is trace-preserving and positive.
2. Write down the Kraus representation for this channel.
3. Calculate the fixed points of this channel, i.e. the states ρ for which $\mathcal{E}(\rho) = \rho$.
4. Solve the master equation corresponding to the purely dissipative Lindbladian (with no Hamiltonian) with the jump operators $\left|0\right\rangle\left\langle +\right|$, $\left|1\right\rangle\left\langle -\right|$ and show that $\mathcal{E} = \lim_{t\to\infty}\mathrm{e}^{\mathcal{L}t}$.

Challenge exercises

Problem 3. Let ρ be a density matrix for two qubits A and B such that the reduced state on qubit A is pure, i.e. $\text{tr}_B(\rho) = |\psi\rangle\langle\psi|$ for some $|\psi\rangle$. Show that

1. If ρ is a pure state, then the reduced state on the qubit B is also pure.
2. ρ is a product state.

Problem 4. Consider the transposition map over $\mathbb{C}^{d\times d}$, i.e. the map that transforms $\rho \in \mathbb{C}^{d\times d}$ to ρ^T. Show that this map is positive, but not completely positive.

Hint: Consider the state

$$\rho_p := p\,|\psi^-\rangle\langle\psi^-| + (1-p)\frac{\mathbb{1}}{4}\,,$$

for $|\psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$*, where* $p \in (0,1)$*, and compute its partial transpose with respect to the second qubit.*