Quantum Information Theory
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Quantum channels and hypothesis testing

Graded exercises

Problem 1. Consider the following state

$$\rho_p = p\,|\Psi^-\rangle\langle\Psi^-| + (1-p)\frac{I}{4}\,,$$

where $p \in (0,1)$, I is the identity operator and

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}\,.$$

1. Show that ρ_p is positive semidefinite for every $p \in (0,1)$.
2. Compute the partial transpose with respect to the second qubit of ρ_p.
3. Using the PPT criterion, calculate the range of p for which the state ρ_p is separable and when it is entangled.

Hint: The PPT criterion states that if ρ_{AB} is separable, then its partial transpose has non-negative eigenvalues. The converse is true for $\rho_{AB} \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ or $\rho_{AB} \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^3)$.

If you need it, you can use the following expression for the partial transpose in A: Consider $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that $\rho_{AB} = \sum_{ijkl} \rho_{ij,kl}\,|i\rangle\langle j|_A \otimes |k\rangle\langle l|_B$. Then,

$$\rho_{AB}^{T_A} = (T \otimes \mathbb{1})(\rho_{AB}) = \sum_{ijkl} \rho_{ij,kl}\,|j\rangle\langle i|_A \otimes |k\rangle\langle l|_B\,.$$

Problem 2. Here we investigate the use of quantum hypothesis testing for distinguishing quantum states.

1. Given the two states $|0\rangle$ and $|+\rangle$, what is the optimal measurement for distinguishing them? Compute the corresponding probability.
2. Now change the second state by $p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$ with $p \in (0,1)$. How does the probability depend on p?
3. In 1., consider now several copies of the states, i.e. $|0000\rangle$ and $|++++\rangle$, for instance. How does the probability of discriminating the states successfully increase with the number of copies?

Problem 3. The fidelity of two mixed states $\rho_A, \sigma_A \in \mathcal{S}(\mathcal{H}_A)$ can be defined by means of their purifications, namely

$$F(\rho_A, \sigma_A) := \max_{|\psi\rangle_{AB}, |\phi\rangle_{AB}} |\langle\psi|\phi\rangle_{AB}|^2 ,$$

where $|\psi\rangle_{AB}$ and $|\phi\rangle_{AB}$ are purifications of ρ_A and σ_A, respectively. Show that the following properties hold:

1. $F(\rho_A, \sigma_A) = 1$ if, and only if, $\rho_A = \sigma_A$.
2. $F(\rho_A, \sigma_A) = 0$ if, and only if, $\rho_A \sigma_A = \sigma_A \rho_A = 0$.
3. For any density operator $\tau \in \mathcal{S}(\mathcal{H}_C)$, we have $F(\rho_A \otimes \tau, \sigma_A \otimes \tau) = F(\rho_A, \sigma_A)$.

Challenge exercises

Problem 4. Let ρ, σ be two density matrices and denote $Q(s) := \mathrm{tr}[\rho^s \sigma^{1-s}]$ for $s \in [0, 1]$, which defines the Chernoff distance by taking

$$-\inf_{s \in [0,1]} \log(Q(s)) .$$

1. Assuming that ρ and σ are pure states, what is the relation between $Q(s)$ and the fidelity $F(\rho, \sigma)$? Remember that, in general,

$$F(\rho, \sigma) := \left(\mathrm{tr}\left[(\rho^{1/2}\sigma\rho^{1/2})^{1/2}\right]\right)^2 .$$

2. Show that

$$\inf_{s \in [0,1]} Q(s) \le F(\rho, \sigma)^{1/2} .$$

3. Show that for every $s \in [0, 1]$,

$$F(\rho, \sigma) \le Q(s) .$$

   Hint: It might help you to write

$$\rho^{1/2}\sigma^{1/2} = \rho^{\frac{1-s}{2}} \left(\rho^{\frac{s}{2}} \sigma^{\frac{1-s}{2}}\right) \sigma^{\frac{s}{2}} .$$

   and use that $\|AB\|_r \le \|A\|_p \|B\|_q$ *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.