Quantum Information Theory
SoSe 2023

Universität Tübingen
Ángela Capel
Paul Gondolf

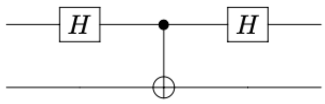
Sheet 8

28. June 2023

Quantum error correction

Graded exercises

Problem 1. Suppose we want to execute the following circuit



where the two input states are arbitrary single qubit states. Since bit flip errors can occur during the circuit, we encode each qubit in the bit flip code.

1. Draw the circuit that executes the above logical transformation when the two qubits are encoded in the bit flip code.
2. Suppose we now want to compute the output of this circuit on the initial state $|0\rangle \otimes |0\rangle$. The encoded initial state is $|000\rangle \otimes |000\rangle$ (where we have only explicitly written the tensor product between blocks). However, suppose a bit flip error occurs on one of the qubits in the encoded block for the first qubit before the first Hadamard, and the erroneous encoded input state becomes $|100\rangle \otimes |000\rangle$. Compute the output of the encoded circuit on this erroneous encoded input state. The error before the circuit will result in error(s) after the circuit. How many errors are there (how many qubits values are flipped from the ideal encoded output state), and are they correctable using the bit flip code detection procedure?

Problem 2. The Shor code is able to protect against phase flip and bit flip errors on any qubit.

1. Show that the syndrome measurement for detecting phase flip errors in the Shor code corresponds to measuring the observables $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$.
2. Show that recovery from a phase flip on any of the first three qubits may be accomplished by applying the operator $Z_1Z_2Z_3$.

Problem 3. Consider the following code:

$$|0\rangle_{en} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),\ |1\rangle_{en} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle).$$

Show that an arbitrary superposition of encoded states $\alpha\,|0\rangle_{en} + \beta\,|1\rangle_{en}$ is robust against errors of the form $\mathrm{e}^{-i\frac{\theta}{2}\sigma_z} \otimes \mathrm{e}^{-i\frac{\theta}{2}\sigma_z}$.

Challenge exercises

Problem 4. Consider a code space $\mathcal{C}$ and errors described by operators $E_1, E_2, \dots E_M$. Suppose that the errors satisfy:

$$\forall\, |\psi\rangle \in \mathcal{C}, \quad \langle \psi \mid E_i^* E_j \psi \rangle = \alpha_i \|\psi\|^2 \delta_{i,j},$$

for some $\alpha_i > 0$.

1. If $\Pi_{\mathcal{C}}$ is the projector on the code space $\mathcal{C}$, show that

$$\Pi_{\mathcal{C}} E_i^* E_j \Pi_{\mathcal{C}} = \alpha_i \Pi_{\mathcal{C}} \delta_{i,j}\,.$$

2. Show that there is a recovery operation that corrects all the errors on a state in the code space.

   Hint: You can follow the ideas used in the proof of Theorem 10.1 of the book of Nielsen and Chuang.