Mock Exam, Mathematical Statistical Physics

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(1) Let $E = \{1, 2, 3, 4, 5\}$ and X_t be a Markov chain with transition matrix´

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1/2 & 1/3 & 1 & 0 & 0 \\ 1/2 & 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Find all classes of this Markov chain and decide, weather they are closed or open. Give one stationary state of that Markov chain.

(2) Let $T : [0, 1] \to [0, 1]$ be the Farey map which is given by:

$$T(x) = \begin{cases} \frac{x}{1-x}, & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1-x}{x}, & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Show that for any interval $I \subset [0, 1]$ it holds that $\mathbb{M}(I) = \mathbb{M}\left(T^{-1}(I)\right)$ preserves the measure given by the density $\frac{1}{x}$, i.e. by

$$\mathbb{M}(A) = \int_A \frac{1}{x} dx \, .$$

Note, that $\mathbb{M}$ is not a probability measure.

(3) Let X be a random variable on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Show that the function $f(x) := \mathbb{P}(x \leq X \leq 0)$ is continuous from the left, i.e. that $\lim_{h \nearrow 0} f(x + h) = f(x)$ for all $x \in \mathbb{R}$. Here $\lim_{h \nearrow 0}$ stands for the limit $h \to 0$ restricted to negative values of h.

(4) Let $T : \Omega \to \Omega$ and $\mathbb{P}$ be a measure on $(\Omega, \mathcal{A})$ that is invariant under T. Let $A \in \mathcal{A}$ be a measurable set with $\mathbb{P}(A) > 0$. As we all know, it follows that the conditional probabilities $\mathbb{P}_A$ and $\mathbb{P}_{A^c}$ are also probability measures (no proof required!). Here A^c stands for the complement of A.

(a) Prove that if T is measure preserving and $T^{-1}(A) = A$ it follows that $\mathbb{P}_A$ and $\mathbb{P}_{A^c}$ are measure preserving.

(b) A probability measure $\mathbb{P}$ is called decomposable with respect to T if for some measurable $A \in \mathcal{A}$ with $\mathbb{P}(A) > 0$.

$$\mathbb{P} = \lambda \mathbb{P}_A + (1 - \lambda) \mathbb{P}_{A^c} \text{ with } \lambda \in]0, 1[$$

Show that if $\mathbb{P}$ is indecomposable with respect to T then T is ergodic (with respect to $\mathbb{P}$).