Exercises: Mathematical Statistical Physics

Prof. Dr. P. Pickl
Manuela Feistl

Sheet 10

Exercise 1: (Maxwell-Boltzmann distribution)
Consider N particles which are distributed among k boxes of equal size. Assume that a particle in box number k has an energy given by E_k and that the total energy is given by E. What density distribution (i.e. what numbers of particles in the different boxes) maximizes the entropy.

Argue that a confined Newtonian system of N particles approaches this distribution.

Hint: Since $N \gg m$ one may assume that $N_j \gg 1$ for all $j = 1, \ldots, m$ and can approximate $N_j! \approx \left(\frac{N_j}{e}\right)^{N_j}$ by Stirling formula. This allows N_j to be treated as continuous variables. Minify $\ln(W_N)$ instead of W_N.

Exercise 2: (Maxwell-Boltzmann distribution II)
Let $f(\vec{x}, \vec{v}, t)$ be particle distribution function with velocity $\vec{v}$, position $\vec{x}$ and time t, and let us define mass density $\rho(\vec{x}, t) := \int_{\mathbb{R}^3} f(\vec{x}, \vec{v}, t)\, d\vec{v}$ and momentum density $\rho(\vec{x}, t) \cdot \vec{u}(\vec{x}, t) := \int_{\mathbb{R}^3} \vec{v} \cdot f(\vec{x}, \vec{v}, t)\, d\vec{v}$. Show the following:

a) Assume a flow of homogeneous particles where each particle has the velocity $\vec{u}$ with uniform distribution $f(\vec{x}, \vec{v}, t) = \rho\, \delta(\vec{v} - \vec{u})$ defined by Delta-Distribution, calculate mass and momentum density.

b) Assume a flow at equilibrium of Maxwell-distribution where the particle velocity are normal-distributed around the mean velocity $\vec{u}$ given by $f(\vec{x}, \vec{v}, t) = \rho \left(\frac{1}{2\pi CT}\right)^{\frac{3}{2}} \exp -\frac{|\vec{v}-\vec{u}|^2}{2CT}$, where C is a constant, and the variance is depending on the temperature T, calculate the mass density.

Exercise 3: (Maxwell-Boltzmann distribution III) Consider a system of N particles with dynamics given by $\dot{x_j} = v_j$ and $\dot{v_j} = F(x_j)$. Assume that $F = \nabla V$ is the gradient of a spherically symmetric potential V which tends to infinity as $|x|$ tends to infinity.

Argue that also on energy-shells the system is not ergodic.

Calculate the density distribution at equilibrium for given angular momentum L under the assumption that the system is ergodic (Hint: Use exercise 1).

Exercise 4: (Recurrence)

(a) Let $X = \mathbb{R}^2$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x, y) = (x + y, y)$. Show that Poincare-recurrence does not hold for this system.

(b) Let $X = \mathbb{T}^2$ and let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be the linear transformation given by $T(x, y) = (x + y \bmod 1, y)$. Here $T = [0, 1]$. Show that *all!* points in any rectangle of the form $[a, b] \otimes [c, d]$ are infinitely recurrent. Hint: consider rational and irrational velocities separately.