Exercises: Mathematical Statistical Physics

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Sheet 11

Exercise 1: (Ergodicity)
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $T : \Omega \to \Omega$ an ergodic measure-preserving transformation. Show that every map which is invariant under T is almost surely constant.

Exercise 2: (Ergodicity)

a) Let $X = [0, 1)$ be the interval with endpoints identified, λ the 1-dimensional Lebesgue measure and $R_\alpha : x \mapsto x + \alpha \mod 1$ be a rational rotation by $\alpha = \frac{p}{q} \in \mathbb{Q}$. Show that R_α is not ergodic with respect to λ by showing that there exists an invariant function $f : X \to R$ for R_α which is not constant λ-almost everywhere.
Hint: Try to use some trigonometric function.

b) Let $\underline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $R_{\underline{\alpha}} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ be the translation on the torus given by $R_{\underline{\alpha}}(x_1, x_2) = (x_1 + \alpha_1 \mod 1, x_2 + \alpha_2 \mod 1)$. Show that if there exists $\underline{n} = (n_1, n_2) \in \mathbb{Z}^2$ such that $\underline{n} = (0, 0)$ and $\langle \underline{n}, \underline{\alpha} \rangle \in \mathbb{Z}$, then $R_{\underline{\alpha}}$ is not ergodic.
Hint: Look for a non-constant invariant trigonometric function $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}$. You can then use it to find a real-valued non-constant invariant function.

Exercise 3: (Probability measure)
Let $(\Omega, \mathcal{A})$ be a measurable space and $T : \Omega \to \Omega$ a transformation. Show the following:

a) If $\mathbb{P}_1, \mathbb{P}_2$ are probability measures on $(\Omega, \mathcal{A})$, then any linear combination $\lambda\mathbb{P}_1 + (1 - \lambda)\mathbb{P}_2$ is also a probability measure $\forall \lambda \in [0, 1]$.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability measure preserved by T and $A \in \mathcal{A}$ with $\mathbb{P}(A) > 0$.

b) $\mathbb{P}_1(B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$ and $\mathbb{P}_2(B) := \frac{\mathbb{P}(A^c \cap B)}{\mathbb{P}(A^c)}$ are defining two measures. Are they also a probability measure?

c) If A is invariant under T (i.e. $T^{-1}(A) = A$), then $\mathbb{P}_1, \mathbb{P}_2$ are invariant (i.e. measure-preserving) under T.

Exercise 4: (Vlasov equaiton)
Consider the following particle system

$$\begin{aligned}\dot{x}_i &= v_i\\ \dot{v}_i &= F(x_i) + \frac{1}{N}\sum_{j\neq i} G(x_i - x_j)\end{aligned}$$

Show that the empirical density

$$f(x,v,t) = \frac{1}{N}\sum_i \delta(x - x_i)\delta(v - v_i)$$

is a weak solution of the Vlasov equation

$$\partial_t f + v\cdot\nabla_x f + F(x)\cdot\nabla_v f + (G * \tilde{f})\cdot\nabla_v f = 0,$$

where $\tilde{f}(x,t) = \int f dv$ and $G * \tilde{f} = G(x-y)\tilde{f}(y)dy$. Consider therefore

$$\sum_{j\neq i} G(x_i - x_j) = \int G(x_i - y)\sum \delta(y - x_j)dx.$$