MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT
DER UNIVERSITÄT TÜBINGEN WiSe 2022/23

Exercises: Mathematical Statistical Physics

Prof. Dr. P. Pickl
Manuela Feistl

Sheet 3

Exercise 1: (Expectation value) Let u be a discrete random variable that can take positive values u_i with probability p_i for $i \in N \subset \mathbb{N}$ and $S := \sum_{i,j} p_i p_j \frac{(u_i - u_j)^2}{u_i u_j}$. Show:

(a) $S = 2\left(\mathbb{E}(u) \cdot \mathbb{E}\left(\frac{1}{u}\right) - 1\right)$

(b) $\mathbb{E}\left(\frac{1}{u}\right) \geq \frac{1}{\mathbb{E}(u)}$

(c) In which case does the equality of the equation in (b) hold?

Exercise 2: (Sigma Algebra) Let $\mathcal{A}_1, \mathcal{A}_2$ be σ -algebras over the same set of Ω.

(a) Give an example where the union $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ -algebra over Ω.

(b) Let $(\sigma_i)_{i \in I}$ be a indexed set of σ-algebras over Ω. Prove that the intersection of σ-algebras $\cap_{i \in I} \sigma_i$ is a σ-algebra.

Exercise 3: (Borel-Cantelli) Let $\{X_n | n \in \mathbb{N}\}$ be a family of Bernoulli random variables with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$ for $\forall n \ \in \mathbb{N}$. How many random variables X_n have the value 1 or 0 if

(a) $p_n = \frac{1}{n^2}$,

(b) $p_n = \frac{1}{n}$ and the family of random variables is independent?

Exercise 4: (Discrete convolution) Let X, Y be independent $\mathbb{Z}$-valued random variables, $(P_X * P_Y)(\{n\}) := \sum_{m=-\infty}^{\infty} P_X(\{m\}) P_Y(\{n - m\})$ be the convolution of the probability measures $P_X(\{n\}) := P(\{X = n\})$ respectively. $P_Y(\{n\}) := P(\{Y = n\})$. Show:

(a) $P_{X+Y} = P_X * P_Y$

(b) Let X, Y be independent and Poisson distributed with parameters μ_X, μ_Y. Calculate and interpret P_{X+Y}.