MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT WiSe 2022/23
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Exercises: Mathematical Statistical Physics

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Sheet 4

Exercise 1: (Sigma Algebra) Show that all open subsets $\mathcal{O} \subseteq \mathbb{R}$ are elements of the Borell Sigma Algebra $\mathcal{B}(\mathbb{R})$.

Exercise 2: (Law of large numbers) Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(X_k)_{k\in\mathbb{N}}$ a sequence of discrete and independent random variables. Furthermore let $\mathbb{E}(X_k^6)$ be finite for each $k \in \mathbb{N}$ and X_k are uniformly distributed with $E(X_k) = 0$. The mean of the first n terms of the sequence are given by $M_n = \frac{1}{n}\sum_{k=1}^{n} X_k$.

a) Prove that

$$\mathbb{E}(|X_k|^i) \leq \mathbb{E}(X_k^6) + 1 \text{ for all } i \in \{1,2,3,4,5\}.$$

b) Prove that there existis a constant $c \in \mathbb{R}$, such that

$$\mathbb{P}(|M_k| > \frac{1}{l}) < \frac{c \cdot l^6}{k^3}[\mathbb{E}(X_1^6) + \mathbb{E}(X_1^6)^2 + 1].$$

Exercise 3: (Almost Sure Convergence) For the random experiment of tossing a fair coin once assume a sequence of random variables $\{X_n | n \in \mathbb{N}\}$ on the sample space $\Omega = \{\text{head}, \text{tail}\}$ are defined as
$X_n(h) = \frac{n}{n+1}$ and $X_n(t) = (-1)^n \; \forall n \in \mathbb{N}$.

(a) Determine whether the resulting real sequence converges or not.

(b) Calculate $P(\{\omega \in \Omega | \lim_{n\to\infty} X_n(\omega) = 1\})$

(c) Does X_n converge to 1 almost surely?