Exercises: Mathematical Statistical Physics

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Sheet 5

Exercise 1: (Convergence in probability)
Show that if $X_n \xrightarrow{d} c$ (converges in distribution), where c is a constant, then $X_n \xrightarrow{p} c$ (converges in probability).
A sequence $X_1, X_2, \ldots$ of real-valued random variables, with cumulative distribution functions $F_1, F_2, \ldots$, is said to converge in distribution to a random variable X with cumulative distribution function F if $\lim_{n\to\infty} F_n(x) = F(x)$, for every number $x \in \mathbb{R}$ at which F is continuous.

Exercise 2: (Terms of convergence)
Let X_n ($\forall n \in \mathbb{N}$) be i.i.d. Uniform$(0, 1)$ random variables and define $Y_n = \min\{X_1, \ldots, X_n\}$. Prove the statements independently of each other:

(a) $Y_n \xrightarrow{d} 0$ (converges in distribution)

(b) $Y_n \xrightarrow{p} 0$ (converges in probability)

(c) $Y_n \xrightarrow{a.s.} 0$ (converges almost surely).

Exercise 3: (Exponential distribution)
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A random variable is called exponentially distributed with parameters $\lambda > 0$ if X is a density of the shape

$$\rho_\lambda(X) = \begin{cases} 0 & x \leq 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$$

with $\lambda \in \mathbb{R}^+$.

(a) Show that ρ_λ is a probability density.

(b) Sketch the distribution function $V_X(t)$.

(c) Compute the mean and variance of X.

Exercise 4: (Convergence in probability and almost sure convergence)
Let Ω be countable (i.e. finite or countably infinite) and $X, X_1, X_2 \dots$ random variables defined on Ω. Show that in this case almost sure convergence follows from convergence in probability (i.e. that in this case convergence in probability and almost sure convergence are equivalent).
Hint: First explain that for all $\omega_i \in \Omega$ with $\mathbb{P}(\omega_i)$ for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\mathbb{P}(|X_n - X| \geq \epsilon) < \mathbb{P}(\omega_i)$ for all $n > N$. Further derive that $\lim_{n\to\infty} X_n(\omega_i) = X(\omega_i)$ for all $\omega_i \in \Omega$ with $\mathbb{P}(\omega_i) > 0$ and use countability from Ω to show that the set of all other ω_i has probability measure zero.