Exercises: Mathematical Statistical Physics

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Sheet 6

Exercise 1: (Heat equation)
Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Show that the convolution $u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}, (t,x) \mapsto u(t,x) = \int_{\mathbb{R}^n} P(t, x-y) g(y)\, dy$, with $P : \mathbb{R}^n \to \mathbb{R}$, the fundamental solution of the heat equation, given by

$$P(t,x) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^n, t > 0 \\ 0 & x \in \mathbb{R}^n, t \le 0, \end{cases}$$

a) is infinitely differentiable ($u \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$),

b) satisfies the heat equation ($\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n :\ u_t(t,x) - \Delta_x u(t,x) = 0$),

c) converges to the initial conditions in the limit ($\forall x_0 \in \mathbb{R}^n, t > 0 : \lim_{(t,x)\to(0,x_0)} u(t,x) = g(x_0)$).

Exercise 2: (Gambelrs Ruin)
Anna and Bodo are playing poker until one of them goes bankrupt. A has capital a, and B bets an amount of $m-a$. In total, m units of money are involved in the game. In each poker round, A and B each bet one unit of money.
A wins each game with probability p. Therefore, B wins with probability $q := 1-p$. We assume that these probabilities are independent of the previous game progress and, in particular, the players' capital.
A is interested in the probability of bankrupting B. Calculate this probability. Additionally, calculate the expected duration of the game until either A or B goes bankrupt.

Exercise 3: (Markov chain)
A two-state Markov chain is given by the transition matrix

$$P(s,t) = (p_{i,j})_{1 \le i,j \le 2} = \begin{pmatrix} 1-s & s \\ t & 1-t \end{pmatrix}$$

for $0 \le s, t \le 1$.

a) Assume $s = t = 1$ and calculate $P^n(1,1)$ for $n \in \mathbb{N}$.
b) Does $p_{i,j}^{(n)}$ converge for $\forall i, j \in \{1,2\}$?

Assume at every time step $n \in \mathbb{N}$ a phone is either free $(= 0)$ or busy $(= 1)$, and it changes these two states according to $P(s,t)$, where the probability of switching its state from busy into free is s and vice versa given by t, therefore staying at its state is $1-s$ and $1-t$, respectively.

c) Assume at time 0 the phone is free, what is the probability that it will stay two time steps free and will be busy at time 3?
d) Assume at time 0 the phone is free, what is the probability that it will be busy at time 3?

Exercise 4: (Markov chain)
Consider the Markov chain defined in exercise 2. Give all communication classes, all absorbing states and all stationary states of that Markov chain. Decide for the different communication classes whether they are closed or not.