Prof. Dr. Roderich Tumulka
Eberhard Karls University Tübingen

Summer semester 2025
April 24, 2025

Mathematical Statistical Physics: Assignment 3

Problem 12: *Normalizing factor of the Gaussian in d dimensions* (hand in, 20 points)
Determine the normalizing factor $\mathscr{N}$ in the expression

$$\rho(\boldsymbol{x}) = \mathscr{N}\ \exp\bigl(-\tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T C^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\bigr)$$

for the density of $\mathcal{N}^d(\boldsymbol{\mu}, C)$, the Gaussian distribution in $\mathbb{R}^d$ with mean $\boldsymbol{\mu}$ and positive definite, symmetric covariance matrix C. Express $\mathscr{N}$ in terms of $\det C$. Use the answer for $d = 1$ without proof.

Problem 13: *Speeds of molecules* (hand in, 20 points)
(a) Determine the most probable value $v_{\max}$ of the speed $v = |\boldsymbol{v}|$ according to the Maxwellian distribution

$$\rho(\boldsymbol{v}) = \mathscr{N}\ \exp\Bigl(-\frac{m|\boldsymbol{v}|^2}{2kT}\Bigr),$$

with given m and T. (*Hint*: The distribution density ρ_v of v is *not* $\mathscr{N}\exp(-mv^2/2kT)$. Why not?)
(b) Is $v_{\max}$ greater or less than $\sqrt{\mathbb{E}(\boldsymbol{v}^2)}$?
(c) Determine $v_{\max}$ for nitrogen gas (N_2), the main constituent of air with $m = 4.6 \cdot 10^{-26}$ kg, at an absolute temperature of $T = 300$ Kelvin.

Problem 14: *Belt theorem* (hand in, 30 points)
Show that for every $\varepsilon, \delta \in (0, 1)$ there is $d_0 \in \mathbb{N}$ such that, for all $d > d_0$, a fraction of at least $1 - \varepsilon$ of the surface area of the unit sphere in $\mathbb{R}^d$ lies within a belt of width 2δ around the equator, $\{\boldsymbol{x} \in \mathbb{R}^d : -\delta < x_1 < \delta, |\boldsymbol{x}| = 1\}$. ("In high dimension, most points on the sphere are near the equator.")

Instructions: Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a random, uniformly distributed point on $\mathbb{S}^{d-1}_1$. Use symmetry to compute the expectation and variance of X_1. Then use the Chebyshev inequality to bound $\mathbb{P}(-\delta < X_1 < \delta)$.

Problem 15: *Marginal of the uniform distribution on the sphere* (hand in, 30 points)
Let u^{d-1}_R be the uniform probability measure on the sphere $\mathbb{S}^{d-1}_R$ of radius $R > 0$ in $\mathbb{R}^d$, and let $(X_1, \ldots, X_d) \sim u^{d-1}_R$. Show that the marginal distribution of $X_1, \ldots, X_k$, $k < d$, has density given by

$$\rho_{k,d,R}(\boldsymbol{x}) = \frac{A_{d-k}}{A_d\, R^{d-2}}\, 1_{\boldsymbol{x}^2 \le R^2}\, \bigl(R^2 - \boldsymbol{x}^2\bigr)^{(d-k)/2-1} \tag{1}$$

with A_d the area of $\mathbb{S}^{d-1}_1$.

Instructions: The marginal $f_k(x_1, \ldots, x_k)$ can also be defined for a non-normalized density function $f(x_1, \ldots, x_d)$,

$$f_k(x_1, \ldots, x_k) = \int dx_{k+1} \cdots dx_d \, f(x_1, \ldots, x_d) \, . \tag{2}$$

Let $\mathbb{B}_r^d$ denote the ball in $\mathbb{R}^d$ of radius r around the origin. Use without proof that for $f = 1_{\mathbb{B}^d_{R+\Delta R}} - 1_{\mathbb{B}^d_R}$,

$$A_d \, R^{d-1} \, \rho_{k,d,R}(\boldsymbol{x}) = \lim_{\Delta R \to 0} f_k(\boldsymbol{x}) / \Delta R \, . \tag{3}$$

Conclude that, for $\boldsymbol{x}^2 < R^2$,

$$\rho_{k,d,R}(\boldsymbol{x}) = \frac{1}{A_d \, R^{d-1}} \frac{\partial}{\partial R} \operatorname{vol}_{d-k} \big\{ \boldsymbol{y} \in \mathbb{R}^{d-k} : \boldsymbol{x}^2 + \boldsymbol{y}^2 \leq R^2 \big\} \, . \tag{4}$$

Hand in: By 23:59 on Monday May 5, 2025 via `urm.math.uni-tuebingen.de`