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Mathematical Statistical Physics: Assignment 4

Problem 16: *Specific heat* (don't hand in)
The heat capacity per mass is called the specific heat capacity or simply the specific heat c. From the relation $\overline{e} = \frac{3}{2}kT$, derive a formula for the specific heat of the ideal gas.

Problem 17: *Net force exerted by pressure* (hand in, 20 points)
Show that in the absence of external fields, the net force $\boldsymbol{K}$ exerted by the pressure of a hard sphere gas according to the Maxwellian distribution on the walls of the container $\Lambda \subset \mathbb{R}^3$ vanishes, regardless of the shape of Λ. (As a consequence, the center of mass of the container does not move if it was at rest initially.)

Instructions: Our derivation of the equation of state, $pV = NkT$, shows, among other things, that the pressure p is constant along the wall. Assume that the boundary $\partial\Lambda$ is piecewise smooth with outward normal vector field $\boldsymbol{n}(\boldsymbol{x})$ for $\boldsymbol{x} \in \partial\Lambda$. The force on the surface element $d^2\boldsymbol{x} \subset \partial\Lambda$ is $p\,\boldsymbol{n}(\boldsymbol{x})\,d^2\boldsymbol{x}$ (where we write $d^2\boldsymbol{x}$ for the infinitesimal set as well as for its area).

Problem 18: *Equivalence of ensembles* (hand in, 40 points)
For an ideal gas without external field in a box Λ, the Hamiltonian is $H(\boldsymbol{q}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{q}_N, \boldsymbol{p}_N) = \sum_{j=1}^N \boldsymbol{p}_j^2/2m$ on the phase space $\Gamma = (\Lambda \times \mathbb{R}^3)^N$. For large N, the micro-canonical distribution μ_{mc} [if you wish with "density" $\rho_{\mathrm{mc}}(x) = \mathscr{N}\,\delta(E - H(x))$] and the canonical distribution μ_{can} [with density $\rho_{\mathrm{can}}(x) = Z^{-1}\exp(-\beta H(x))$] are not very different, provided $E = E(\beta)$ [or $\beta = \beta(E)$] is suitably chosen: Both are constant on every energy surface, and both are narrowly concentrated around a certain energy value. To see this, proceed as follows.

(a) Show that $Z = \frac{1}{2}\,\mathrm{vol}(\Lambda)^N A_{3N} (2m/\beta)^{3N/2} \Gamma(3N/2)$.

(b) For $X \sim \mu_{\mathrm{can}}$, determine $\mathbb{E}H(X)$ and $\mathrm{Var}\,H(X)$. [*Hint*: $\Gamma(x+1) = x\Gamma(x)$.]

(c) Which relation $E(\beta)$ is required to ensure that μ_{mc} and μ_{can} have the same expected energy?

(d) How large is $\mathrm{Var}\,H(X)$ compared to $[\mathbb{E}H(X)]^2$?

Problem 19: *A variant of the belt theorem* (hand in, 20 points)
Let $\boldsymbol{X}, \boldsymbol{Y}$ be independent, uniformly distributed random unit vectors in $\mathbb{R}^d$. Show that for every $\varepsilon, \delta \in (0,1)$ there is $d_0 \in \mathbb{N}$ such that, for all $d > d_0$, $|\boldsymbol{X} \cdot \boldsymbol{Y}| < \delta$ with probability at least $1 - \varepsilon$. ("In high dimension, independent purely random unit vectors are nearly orthogonal.")
Hint: Conditionalize on $\boldsymbol{X}$ and use the belt theorem.

Problem 20: *Again near-orthogonality in high dimension* (hand in, 20 points)
Let $\boldsymbol{X}, \boldsymbol{Y}$ be independent, uniformly distributed random unit vectors in $\mathbb{R}^d$.

(a) Explain why $\mathbb{E}[Y_1^2] = 1/d$.

(b) Explain why $\mathbb{E}[(\boldsymbol{X} \cdot \boldsymbol{Y})^2] = 1/d$.

(c) Consider $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}_1^{d-1}$ with $(\boldsymbol{x} \cdot \boldsymbol{y})^2 = 1/d$. Compute the angle α between $\boldsymbol{x}$ and $\boldsymbol{y}$ for $d = 3$ and asymptotically for large d (to leading non-constant order in d).

Hand in: By 23:59 on Monday, May 12, 2025 via `urm.math.uni-tuebingen.de`