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Mathematical Statistical Physics: Assignment 5

Problem 21: *Momentum reversal* (don't hand in)
Let $R = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ be the diagonal matrix with n ones and n minus-ones, so $R\binom{q}{p} = \binom{q}{-p}$. Assume that $H(Rx) = H(x)$ for all $x \in \Gamma = \mathbb{R}^{2n}$. Derive from Hamilton's equations of motion (2.17)–(2.18) that $T^{-t}x = RT^t(Rx)$ for all $x \in \Gamma$ and $t \in \mathbb{R}$, assuming global existence (and uniqueness) of solutions and $H \in C^2$.

Problem 22: *Poincaré recurrence* (hand in, 40 points)
Let $\Omega = \mathbb{S}^1_1$ be the unit circle in $\mathbb{R}^2 = \mathbb{C}$, and let $T : \Omega \to \Omega$ be multiplication by $e^{i\alpha}$. Use the Poincaré recurrence theorem to show:

(a) $\forall \delta > 0 : \exists n \in \mathbb{N} : \forall x \in \Omega : \ d(x, T^n x) < \delta,$
where d is the distance in arc length along the circle.

(b) For $\alpha \notin \pi\mathbb{Q}$ and every $x \in \Omega$, the set $T^{\mathbb{N}}x$ is dense in Ω.

Problem 23: *Dense trajectory* (hand in, 30 points)
Let Ω be the 2-dimensional torus and φ_1 and φ_2 the angular coordinates on it (longitude and latitude). For given constants $\alpha_1, \alpha_2 \in \mathbb{R}$, consider the ODE

$$\frac{d\varphi_1}{dt} = \alpha_1\,, \quad \frac{d\varphi_2}{dt} = \alpha_2\,.$$

(a) Give an explicit formula for the flow map: $T^t(\varphi_1, \varphi_2) = ?$

(b) Use Problem 22 to show that if $\alpha_2 \neq 0$ and $\alpha_1/\alpha_2 \notin \mathbb{Q}$, then the curve $t \mapsto T^t(\varphi_1, \varphi_2)$ is dense on the torus.

Problem 24: *Dense trajectory in higher dimension* (don't hand in)
Consider the corresponding situation on the n-dimensional torus $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$:

$$\frac{d\varphi_i}{dt} = \alpha_i\,, \quad i = 1, \ldots, n\,.$$

Under which condition on $(\alpha_1, \ldots, \alpha_n)$ is the curve $t \mapsto T^t(\varphi_1, \ldots, \varphi_n)$ dense on the torus?

Problem 25: *Recurrence times* (don't hand in)
In order to estimate the order of magnitude of realistic recurrence times, we reason as follows. An ideal gas comprising $N = 10^{23}$ particles (or a gas of 10^{23} hard spheres, the difference does not matter) in a box Λ starts in such a phase point x_0 that all particles are located in the left half Λ_L of the box; apart from that, let x_0 be typical of energy $E = N\bar{e}$; i.e., take x_0 to be a typical element of $M_L = \Gamma_E \cap (\Lambda_L^N \times \mathbb{R}^{3N})$. We want to know how long it takes, after $x(t)$ has left M_L, until $x(t)$ returns to M_L.

(a) Determine $\mu_E(M_L)$.

(b) Think of Γ_E as partitioned into cells $C_1, \ldots, C_r$ of equal volume (i.e., of equal measure μ_E), of which M_L is one. Assume that every cell gets traversed in time τ, and that the trajectory $x(t)$ visits all cells in a random-looking order. How many years will pass before the return to M_L if $\tau = 10$ s? If $\tau = 10^{-20}$ s?

Problem 26: *Scattering cross section for billiard balls* (hand in, 30 points)
When two billiard balls of radius a and momenta $\boldsymbol{p}_1, \boldsymbol{p}_2$ collide, the resulting (outgoing) momenta $\boldsymbol{p}_1', \boldsymbol{p}_2'$ depend on the displacement vector $\boldsymbol{\omega} = (\boldsymbol{q}_2 - \boldsymbol{q}_1)/2a \in \mathbb{S}_1^2$ at the time of the collision:

$$\boldsymbol{p}_1' = \boldsymbol{p}_1 - [(\boldsymbol{p}_1 - \boldsymbol{p}_2) \cdot \boldsymbol{\omega}]\boldsymbol{\omega}\,, \qquad \boldsymbol{p}_2' = \boldsymbol{p}_2 + [(\boldsymbol{p}_1 - \boldsymbol{p}_2) \cdot \boldsymbol{\omega}]\boldsymbol{\omega}\,. \tag{1}$$

We consider *random* collisions and want to characterize the probability distribution of $\boldsymbol{p}_1', \boldsymbol{p}_2'$ for given $\boldsymbol{p}_1, \boldsymbol{p}_2$ by determining that of $\boldsymbol{\omega}$. To this end, we suppose that $\boldsymbol{p}_2 = \boldsymbol{0}$ (as can be arranged via a Galilean transformation), $\boldsymbol{q}_2 = \boldsymbol{0}$, and $\boldsymbol{p}_1 = p_1 \boldsymbol{e}_x$ (via translation and rotation). It is reasonable to assume that the y- and z-components of $\boldsymbol{q}_1$ are uniformly distributed on the disc of radius $2a$ around the origin in the yz-plane (given that a collision occurs at all); the polar coordinates r and φ of (y, z) are called the collision parameters.

(a) Express $\boldsymbol{q}_1$ and $\boldsymbol{\omega}$ as functions of r and φ.
(b) Show that $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ has probability density proportional to $1_{\omega_x < 0}\, |\omega_x|$ relative to the uniform measure $u(d^2\boldsymbol{\omega})$ on the sphere.
(c) Explain why, for arbitrary $\boldsymbol{p}_1, \boldsymbol{p}_2$, the probability distribution of $\boldsymbol{\omega}$ is proportional to $1_{\boldsymbol{\omega}\cdot(\boldsymbol{p}_1 - \boldsymbol{p}_2) < 0} \left|\boldsymbol{\omega} \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_2)\right| d^2\boldsymbol{\omega}$.

Hand in: By 23:59 on Monday, May 19, 2025 via `urm.math.uni-tuebingen.de`