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Mathematical Statistical Physics: Assignment 6

Problem 27: *Refrigerator* (don't hand in)
Use the second law of thermodynamics to show that a refrigerator necessarily consumes energy rather than generating energy. (We might have thought that the energy removed from the content of the refrigerator is available afterwards.)

Instructions. Let T_{in} be the temperature inside the refrigerator, T_{out} the one outside, Q_{in} the heat energy inside, Q_{out} the one outside, and δW the usable energy provided by the refrigerator (negative if it consumes energy) while it adds the energy $\delta Q_{\text{in}} < 0$ to the content and δQ_{out} to the outside. Use the Clausius relation $\delta S_i = \delta Q_i / T_i$, $i = \text{in}, \text{out}$ to show that $\delta W < 0$.

Problem 28: *Entropy in thermal equilibrium* (hand in, 25 points)
Compute the entropy $S(E, V, N)$ of the thermal equilibrium state of an ideal mono-atomic gas from Boltzmann's formula $S(\text{eq}) = k \log \Omega(E)$.

Instructions. We have already found the relations

$$\operatorname{vol} \Gamma_{\leq E} = \frac{1}{N!} V^N \, V_{3N} (2mE)^{3N/2} \tag{1}$$

$$V_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)} \tag{2}$$

$$\Omega(E) = \frac{d}{dE} \operatorname{vol} \Gamma_{\leq E} \,. \tag{3}$$

Use Stirling's formula

$$\Gamma(x+1) = \sqrt{2\pi x}\, e^{-x} x^x (1 + o(1)) \quad \text{as } x \to \infty \tag{4}$$

and $n! = \Gamma(n+1)$. Set $E = Ne$ and $V = Nv$ with constants e, v, sort terms by orders $O(N \log N), O(N), O(\log N), \ldots$, and give the leading order terms as the answer.

Problem 29: *Concave function* (hand in, 20 points)
Verify that the function

$$S(E, V, N) = kN \log \frac{V}{N v_0} + \tfrac{3}{2} kN \log \frac{E}{N e_0}$$

is concave on $(0, \infty)^3$. Use without proof that a C^2 function is concave if and only if its Hessian is everywhere negative semi-definite. (*Hint*: The matrix is singular, and on a certain 2d subspace it is negative definite.)

Problem 30: *Ergodicity on a finite set* (hand in, 35 points)
Let Ω be a finite set, $\#\Omega = n$. A dynamical system in discrete time means a bijection $T : \Omega \to \Omega$.
(a) Show that T preserves the uniform measure $\mathbb{P}(A) = \#A/n$.
(b) Show that $\mathbb{P}$ is the only probability measure preserved by every bijection.
(c) How many dynamical systems on Ω exist?
(d) What should it mean here for T to be ergodic?
(e) What are the ergodic components of a non-ergodic T?
(f) How many dynamical systems on Ω are ergodic?
(g) Determine the probability that a randomly chosen dynamical system on Ω is ergodic.
(h) The recurrence time of $\omega \in \Omega$ for T is defined as the smallest $t \in \mathbb{N}$ with $T^t\omega = \omega$. Determine the recurrence time for ergodic T.
(i) Determine the average recurrence time for random T (ergodic or not) and random ω.

Problem 31: *Gibbs entropy* (hand in, 20 points)
The Gibbs entropy of a probability density function ρ on phase space $\Gamma = \mathbb{R}^d$ is defined by

$$S_{\mathrm{Gibbs}}(\rho) = -k \int_\Gamma dx\, \rho(x)\, \log \rho(x) \tag{5}$$

with the convention $0 \log 0 := 0$. Suppose $M : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism with Jacobian determinant $|\det DM(x)| = 1$ at all $x \in \mathbb{R}^d$ (so M preserves volumes), and suppose that the point X_0 in $\mathbb{R}^d$ is chosen randomly with (smooth) probability density $\rho_0 : \mathbb{R}^d \to [0, \infty)$. Use the transformation formula for integrals to show that

(a) the random point $Y = M(X_0)$ has density $\rho_1(y) = \rho_0(M^{-1}(y))$.

(b) $S_{\mathrm{Gibbs}}(\rho_1) = S_{\mathrm{Gibbs}}(\rho_0)$.

Remark. As a consequence, for a Hamiltonian system on $\mathbb{R}^d$ such that for each $t \in \mathbb{R}$ the flow map T^t is a diffeomorphism $\mathbb{R}^d \to \mathbb{R}^d$, if ρ_t is the density of $X_t = T^t(X_0)$ then, since T^t has Jacobian determinant 1 (Liouville's theorem), the Gibbs entropy never changes with time.

Hand in: By 23:59 on Monday, May 26, 2025 via `urm.math.uni-tuebingen.de`