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Mathematical Statistical Physics: Assignment 9

Problem 38: *Moments of a random wave function* (hand in, 50 points)
Let $\mathbb{S}$ be the unit sphere in the Hilbert space $\mathbb{C}^d$, u the uniform probability distribution on $\mathbb{S}$, and $\Psi = (\Psi_1, \ldots, \Psi_d) \sim u$. Compute all moments of Ψ of up to fourth order. That is, show for all $k, \ell, m, n \in \{1, \ldots, d\}$ that
(a) $\mathbb{E}\Psi_k = 0$ (*Hint*: symmetry)
(b) $\mathbb{E}\Psi_k^*\Psi_\ell = 0 = \mathbb{E}\Psi_k\Psi_\ell$ for $k \neq \ell$
(c) $\mathbb{E}|\Psi_k|^2 = 1/d$
(d) $\mathbb{E}\Psi_k^2 = 0$
(e) $\mathbb{E}\Psi_k\Psi_\ell\Psi_m = 0$, and likewise if any of the factors is conjugated
(f) $\mathbb{E}\Psi_k\Psi_\ell\Psi_m\Psi_n = 0$ if an index occurs only once, and likewise for conjugated factors
(g) $\mathbb{E}\Psi_k^4 = 0 = \mathbb{E}\Psi_k^{*4} = \mathbb{E}\Psi_k^*\Psi_k^3 = \mathbb{E}\Psi_k^{*3}\Psi_k$
(h) $\mathbb{E}|\Psi_k|^4 = \dfrac{2}{d(d+1)}$ (the main problem!)

(*Instructions*: Regard $\mathbb{C}^d$ as $\mathbb{R}^{2d}$, $\Psi = (x_1, \ldots, x_{2d}) = \boldsymbol{x}$, $I_1 = \int_{\mathbb{S}} u(d\boldsymbol{x})\, x_1^4$, $I_2 = \int_{\mathbb{S}} u(d\boldsymbol{x})\, x_1^2 x_2^2$. Integrating in spherical coordinates,[1]

$$\int_{\mathbb{R}^{2d}} d\boldsymbol{x}\, x_1^2\, x_2^2 \exp(-|\boldsymbol{x}|^2) = \int_0^\infty dr\, r^{2d-1}\, r^4\, \exp(-r^2)\, I_2\, \mathrm{area}(\mathbb{S})\,. \tag{1}$$

Now the substitution $s = r^2$ helps. Use without proof that $\mathbb{E}\big[(X-\mu)^4\big] = 3\sigma^4$ for $X \sim \mathcal{N}(\mu, \sigma^2)$.)

(i) $\mathbb{E}|\Psi_k|^2|\Psi_\ell|^2 = \dfrac{1}{d(d+1)}$ for $k \neq \ell$ (*Hint*: $\mathbb{E}\big[(\sum_k |\Psi_k|^2)^2\big] = 1$ (why?).)
(j) $\mathbb{E}\Psi_k^2\Psi_\ell^2 = 0 = \mathbb{E}|\Psi_k|^2\Psi_\ell^2 = \mathbb{E}\Psi_k^{*2}\Psi_\ell^2$ for $k \neq \ell$.

Problem 39: *Variance and covariance of a random wave function* (hand in, 25 points)
(a) For Ψ as in Problem 38, conclude from the results of Problem 38 that

$$\mathrm{Var}\big(|\Psi_1|^2\big) = \frac{1}{d^2}\frac{d-1}{d+1}\,, \quad \mathrm{Cov}\Big(|\Psi_1|^2, |\Psi_2|^2\Big) = -\frac{1}{d^2(d+1)}\,.$$

(b) As we know, for large d, Ψ_1 is approximately $\mathcal{N}^2(\mathbf{0}, I/2d)$ distributed. For comparison, let $\boldsymbol{G} = (G_1, \ldots, G_d) = (X_1, \ldots, X_{2d})$ be a Gaussian random vector in $\mathbb{C}^d = \mathbb{R}^{2d}$, i.e., so that the X_i (the real and imaginary parts of the G_k) are i.i.d. with $X_i \sim \mathcal{N}(0, 1/2d)$. Determine $\mathrm{Var}(|G_1|^2)$ and $\mathrm{Cov}(|G_1|^2, |G_2|^2)$.

[1] This trick was discovered by N. Ullah, *Nuclear Physics* **58**: 65–71 (1964).

Problem 40: *Quantum particle in a box in 1d* (don't hand in)
(a) On the interval $[0, L]$, consider the Hamiltonian operator $H\psi(x) = -\psi''(x)/2m$ with Dirichlet boundary conditions $\psi(0) = 0$, $\psi(L) = 0$. Verify that the normalized eigenfunctions read

$$\varphi_n(q) = \left(\frac{2}{L}\right)^{1/2} \sin(n\tfrac{\pi}{L}q) \tag{2}$$

with $n \in \mathbb{N}$ and eigenvalues

$$E_n = \frac{\pi^2}{2mL^2}\, n^2\,. \tag{3}$$

(b) It is known from Fourier series that the functions $1, \sin nx, \cos nx$ $(n \in \mathbb{N})$, after normalization, form an orthonormal basis of $L^2([-\pi, \pi])$. How can we conclude that the functions (2) form an orthonormal basis of $L^2([0, L])$?

Problem 41: *Projection to fermionic wave functions* (hand in, 25 points)
We write $(-1)^\sigma$ for the sign of a permutation $\sigma \in S_N$. We want to show that P_- defined by

$$P_-\psi(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma\, \psi(\boldsymbol{q}_{\sigma(1)}, \ldots, \boldsymbol{q}_{\sigma(N)}) \tag{4}$$

is the orthogonal projection to the subspaces of anti-symmetric functions in $\mathscr{H} = L^2(\mathbb{R}^{3N})$. Proceed as follows:
(a) $P_-\psi$ is an anti-symmetric function.
(b) If ψ is already anti-symmetric, then $P_-\psi = \psi$.
(c) $P_-^2 = P_-$
(d) $P_- : \mathscr{H} \to \mathscr{H}$ is self-adjoint.
From (c) and (d) it follows that P_- is an orthogonal projection, and from (a) and (b) that its range is the space of anti-symmetric functions in $\mathscr{H}$.

Hand in: By 23:59 on Monday, July 7, 2025 via `urm.math.uni-tuebingen.de`