Mathematical Statistical Physics: Assignment 10

Problem 42: *Complex Gaussian in d dimensions* (hand in, 25 points)
The complex random variable X obeys the *complex Gaussian distribution* $\mathcal{N}_{\mathbb{C}}(z,\sigma^2)$ whenever $\operatorname{Re} X \sim \mathcal{N}(\operatorname{Re} z, \sigma^2/2)$, $\operatorname{Im} X \sim \mathcal{N}(\operatorname{Im} z, \sigma^2/2)$, and $\operatorname{Re} X$ and $\operatorname{Im} X$ are independent. The *Gaussian distribution* $\mathcal{N}_{\mathbb{C}^d}(\psi_0, C)$ *on* $\mathbb{C}^d$ has density of the form

$$f(\psi) = \mathscr{N}\,\exp\Bigl(-\langle \psi-\psi_0|C^{-1}|\psi-\psi_0\rangle\Bigr)\,, \tag{1}$$

where the complex $d\times d$ matrix C is self-adjoint and positive definite. Use what we know about the real Gaussian distribution to show for $\Psi \sim \mathcal{N}_{\mathbb{C}^d}(\psi_0, C)$ that
(a) $\mathscr{N} = \pi^{-d}(\det C)^{-1}$
(b) C is the complex covariance matrix, $C = \mathbb{E}|\Psi-\psi_0\rangle\langle\Psi-\psi_0|$ (*hint*: diagonalize)
(c) for every $\phi\in\mathbb{C}^d$ we have that $\langle\phi|\Psi\rangle$ obeys a complex Gaussian distribution.

Problem 43: *Conditional wave function* (hand in, 25 points)
For a given ONB $\{\phi_q\}$ of $\mathscr{H}_b$ and $\psi \in \mathbb{S}(\mathscr{H}_s\otimes\mathscr{H}_b)$, the *conditional wave function* ψ_s is the random vector in $\mathbb{S}(\mathscr{H}_s)$ given by

$$\psi_s = \mathscr{N}\,\langle\phi_Q|\psi\rangle_b$$

with partial inner product only in b, normalizing factor $\mathscr{N}$, and random Q,

$$\mathbb{P}(Q=q) = \|\langle\phi_q|\psi\rangle_b\|_s^2\,.$$

Show that the density matrix of the distribution μ_s of ψ_s is exactly the reduced density matrix of ψ, $\rho_{\mu_s} = \rho_s^\psi$ with $\rho_{\mu_s} = \mathbb{E}|\psi_s\rangle\langle\psi_s|$ and $\rho_s^\psi = \operatorname{tr}_b|\psi\rangle\langle\psi|$.

Problem 44: *Typicality of macroscopic thermal equilibrium* (hand in, 25 points)
Let $0<\varepsilon<\delta<1$. Let $\mathscr{H}_{\mathrm{eq}}$ be a subspace of the finite-dimensional Hilbert space $\mathscr{H}_{\mathrm{mc}}$ with $\dim\mathscr{H}_{\mathrm{eq}}/\dim\mathscr{H}_{\mathrm{mc}} = 1-\varepsilon$, let P_{eq} be the projection to $\mathscr{H}_{\mathrm{eq}}$, and let u_{mc} be the normalized uniform distribution over $\mathbb{S}(\mathscr{H}_{\mathrm{mc}})$. Show for the set

$$\mathrm{MATE} = \Bigl\{\psi\in\mathbb{S}(\mathscr{H}_{\mathrm{mc}}) : \langle\psi|P_{\mathrm{eq}}|\psi\rangle > 1-\delta\Bigr\}$$

that

$$u_{\mathrm{mc}}(\mathrm{MATE}) > 1-\frac{\varepsilon}{\delta}\,.$$

(As a consequence, if $\varepsilon\ll\delta\ll 1$, most wave functions lie in the set MATE of "macroscopic thermal equilibrium states.") *Hint*: Determine the average of $\langle\psi|P_{\mathrm{eq}}|\psi\rangle$ on $\mathbb{S}(\mathscr{H}_{\mathrm{mc}})$. If a function $f\leq 1$ has average near 1, why does it have to be close to 1 at most points?

Problem 45: *Eigenfunctions in equilibrium* (hand in, 25 points)
Show that most eigenvectors of H in $\mathscr{H}_{\mathrm{mc}}$ (in fact at least the fraction $1-\varepsilon/\delta$) lie in the set MATE from Problem 44. (Actually, that is true of every ONB. *Hint*: Proceed analogously to Problem 44.)

Hand in: By 23:59 on Monday, July 14, 2025 via `urm.math.uni-tuebingen.de`