Prof. Dr. R. Tumulka
Mathematisches Institut, Universität Tübingen

Summer semester 2025
July 2025

# Practice Exam Math. Stat. Phys.

**Instructions:**

- These problems are similar in style to the exam problems. Don't hand them in, they will not be graded.
- The material covered by the exam comprises Sections 1–11.5 of the lecture notes and all homework assignments.
- The exam takes 90 minutes. It takes place on Monday July 28, 2025, at 9:15-10:45 am in room N14.
- During the exam, it is not allowed to use books, electronic devices, or notes from before the exam.

**Problem 1:** (10 points) State Liouville's theorem.

**Problem 2:** (10 points) State the definition of ergodicity or an equivalent condition.

**Problem 3:** (10 points) Explain what kind of statement a "validity theorem" of the Boltzmann equation is.

**Problem 4:** (10 points) Explain Zermelo's "recurrence" objection to any attempted derivation of the second law from mechanics. (No need to answer the objection.)

**Problem 5:** (12 points) Consider a random point $\boldsymbol{X} = (X_1, \ldots, X_d)$ with uniform distribution $u_R$ over the sphere $\mathbb{S}_R^{d-1}$ of radius $R = \sqrt{d}$ in $\mathbb{R}^d$. State mathematically what is meant by saying that
**(a)** "the marginal of $u_R$ is Gaussian for large $d$";
**(b)** "the empirical distribution of $X_1, \ldots, X_d$ is typically Gaussian for large $d$."

**Problem 6:** (10 points) The integral

$$\mathscr{I} := \int_{\mathbb{R}^d} d^d\boldsymbol{x}\; e^{-\boldsymbol{x}^2} \tag{1}$$

can be computed in two ways: as a product of $d$ 1-dimensional integrals (whose values we know), or in spherical coordinates (where the angle integrals yield the area $A_d$ of $\mathbb{S}_1^{d-1}$). Exploit this to show that

$$A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,. \tag{2}$$

**Problem 7:** (14 points) Use the Poincaré recurrence theorem to show that for the discrete-time dynamical system on the unit circle $\mathbb{S}^1_1 = \{z \in \mathbb{C} : |z| = 1\}$ given by $Tz = e^{i\alpha}z$, the set $\{T^n 1 : n \in \mathbb{N}\}$ is dense on the circle if $\alpha/\pi$ is irrational.

**Problem 8:** (12 points) We want to show that $P_+$ defined by

$$P_+\psi(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N) = \frac{1}{N!} \sum_{\sigma \in S_N} \psi(\boldsymbol{q}_{\sigma(1)}, \ldots, \boldsymbol{q}_{\sigma(N)}) \tag{3}$$

is the orthogonal projection to the subspace of symmetric (bosonic) functions in $\mathscr{H} = L^2(\mathbb{R}^{3N})$. Proceed as follows:

**(a)** $P_+\psi$ is a symmetric function.
**(b)** If $\psi$ is already symmetric, then $P_+\psi = \psi$.
**(c)** $P_+^2 = P_+$
**(d)** $P_+ : \mathscr{H} \to \mathscr{H}$ is self-adjoint.

**Problem 9:** (12 points) Consider the quantum system consisting of $N$ non-interacting spins, $\mathscr{H} = (\mathbb{C}^2)^{\otimes N}$, $H = \sum_{i=1}^N I \otimes \cdots \otimes I \otimes H_i \otimes I \otimes \cdots \otimes I$ with $H_i$ in the $i$-th place given by

$$H_i = \begin{pmatrix} 1/\sqrt{N} & 0 \\ 0 & -1/\sqrt{N} \end{pmatrix}.$$

Find the density of states $\Omega(E)$ in the limit $N \to \infty$.

**—THE END—**